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Higher Lie and Leibniz algebras

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1 Introduction

Higher structures – infinity algebras and other objects up to homotopy, categorified algebras, 'oidified' concepts, operads, higher categories, higher Lie theory, higher gauge theory ... – are currently intensively investigated in Mathematics and Physics.

The present thesis deals with abstractions and flexibilizations of algebraic structures, and more precisely, with algebraic operads, homotopification and categorification.

1.1 Algebraic operads

Operads can be traced back to the fifties and sixties – let us mention the names of Boardman, MacLane, Stasheff, Vogt ... They were formally introduced by J. P. May in [May72], who also suggested the denomination 'operad' (contraction of 'operation' and 'monad'). Operads are to algebras, what algebras are to matrices, or, better, to representations. More precisely, an operad encodes a type of algebra. It heaves the algebraic operations of the considered type, their symmetries, their compositions, as well as the specific relations they verify, on a more abstract and universal level, which is best thought of by viewing a universal abstract operation as a kind of tree with a finite number of leaves (or inputs) and one root (or output). To 'any' type of algebra one can associate an operad; the representations of this operad form a category, which is equivalent to the category of algebras of the considered type.

Operads were initially studied as a tool in homotopy theory, but found some thirty years later interest in a number of other domains like homological algebra, category theory, algebraic geometry, mathematical physics ... Among various powerful aspects of operads, let us mention that the operadic language simplifies not only the formulation of the mathematical results but also their proofs, that it allows to gain a more conceptual and deeper insight into classical theorems and to extend them to other types of algebra ... ; e.g. if some construction is possible 'mutatis mutandis' for several types of algebra, it is a very enriching challenge to prove that it goes through for operads.

1.2 Homotopy algebras

Homotopy, sh, or infinity algebras [Sta63] are homotopy invariant extensions of differential graded algebras. They appear for instance when examining whether a compatible algebraic structure on some chain complex can be transferred to a homotopy equivalent complex (V, d_V). For differential graded associative or Lie algebras, the naturally constructed product on V is no longer associative or Lie, but verifies the associativity or Jacobi condition up to homotopy. More precisely, we obtain a sequence $m_n : V^{\times n} \to V$, $n \in \mathbb{N}^*$, of multilinear maps on the graded vector space V that (have degree $n-2$ and possibly some symmetry properties) verify a whole sequence of defining relations. We refer to these data as a (an associative or Lie) homotopy algebra structure on V.

Their homotopy invariance, explains the origin of infinity algebras in BRST formalism of closed string field theory. One of the prominent applications of Lie infinity algebras (L_∞-algebras) [LS93] is their appearance in Deformation Quantization of Poisson manifolds. The deformation map can be extended from differential graded Lie algebras (DGLAs) to L_∞-algebras and more precisely to a functor from the category $\mathtt{L}_\infty$ to the category $\mathtt{Set}$. This functor transforms a weak equivalence into a bijection. When applied to the DGLAs of polyvector fields and polydifferential operators, the latter result, combined with the formality theorem, provides the 1-to-1 correspondence between Poisson tensors and star products.

Homotopy Lie and Leibniz algebras are central concepts of the present text.

1.3 Categorified algebras

- Categorification [CF94], [Cra95] is characterized by the replacement of sets (resp., maps, equations) by categories (resp., functors, natural transformations). Rather than considering two maps as equal, one details a way of identifying them. Categorified Lie algebras were introduced by J. Baez and A. Crans [BC04] under the name of Lie 2-algebras. Such an algebra is made up by a category L, a functor $[-,-] : L \times L \to L$, and a natural transformation $J_{x,y,z} : [x,[y,z]] \to [[x,y],z] + [y,[x,z]]$ that verifies some coherence law. The generalization of Lie 2-algebras, weak Lie 2-algebras, was studied by D. Roytenberg [Roy07]. Other generalizations, Leibniz 2-algebras and Lie 3-algebras, are part of the present work.

Categorification is thus a sharpened viewpoint that leads to astonishing results in TFT, bosonic string theory ... Considerable effort with convincing output has been made by J. Baez and his school to show that categorification leads to a deeper understanding of Physics.

- The preceding 'vertical' categorification has a 'horizontal' counterpart: if a notion can be interpreted as some category with a single object, it might be generalized, horizontally categorified, or oidified, by considering many object categories of the same type. Well-known examples of this process are e.g. Lie groupoids and Lie algebroids. The concept of groupoid was introduced in 1926 by H. Brandt; topological and differential groupoids go back to C. Ehresmann and are transitive Lie groupoids in the sense of A. Kumpera and A. Weinstein. Lie algebroids were first considered by J. Pradines [Pra67], following works by C. Ehresmann and P. Libermann.

Lie groupoids and algebroids are of importance in particular in view of their relations with Poisson Geometry and the theory of connections. A generalization of Lie algebroids to the Leibniz case can be found in this work.

1.4 Structure and main results

This dissertation is divided into five parts: a survey work on the operadic approach to homotopy algebras, three research papers, as well as an Appendix containing additional information.

Operadic approach to homotopy algebras

After the introduction of homotopy algebras, it was understood quite quickly that the maps m_n of a homotopy associative or Lie algebra on a space V can be viewed as the corestrictions of a coderivation on the free graded associative or commutative coalgebra over the suspended space sV and that, more surprisingly, the above-mentioned sequence of defining relations can be encrypted in the unique requirement that this coderivation be a codifferential: a (an associative or Lie) homotopy algebra can be interpreted as a codifferential of an appropriate coalgebra.

In their famous paper on 'Koszul duality for operads' [GK94], V. Ginzburg and M. Kapranov gave a conceptual approach to a broad family of homotopy algebras and extended the preceding interpretation to any type of homotopy algebra whose corresponding algebra type can be encoded in a so-called Koszul operad. This operadic approach to homotopy algebras is the main aspect of the survey part of this thesis.

Higher categorified algebras versus bounded homotopy algebras

The work 'Higher categorified algebras versus bounded homotopy algebras' (vertical categorification and homotopification of Lie algebras) was transmitted by J. Stasheff to the journal 'Theory and Applications of Categories' – Theo. Appl. Cat., 25(10) (2011), 251-275.

A categorical definition of Lie 3-algebras is given and their 1-to-1 correspondence with 3-term Lie infinity algebras whose bilinear and trilinear maps vanish in degree $(1,1)$ and in total degree 1, respectively, is proven. Further, an answer to a question of Roytenberg pertaining to the use of the nerve and normalization functors in the study of the relationship between categorified algebras and truncated sh algebras, is provided.

We refer the reader to the introduction of the paper for more details – see page 50.

The Supergeometry of Loday Algebroids

The paper 'The Supergeometry of Loday Algebroids' (horizontal categorification of Leibniz (Loday) algebras) has recently been published in the 'Journal of Geometric Mechanics' – J. Geo. Mech., 5(2) (2013), 185-213 (American Institute of Mathematical Sciences).

A new concept of Loday algebroid (and its pure algebraic version – Loday pseudoalgebra) is proposed and discussed in comparison with other similar structures present in the literature. The structure of a Loday pseudoalgebra and its natural reduction to a Lie pseudoalgebra are studied. Further, Loday algebroids are interpreted as homological vector fields on a 'supercommutative manifold' associated with a shuffle product, and the corresponding Cartan calculus is introduced. Several examples, including Courant algebroids, Grassmann-Dorfman and twisted Courant-Dorfman brackets, as well as algebroids induced by Nambu-Poisson structures, are given.

For further details, we refer the reader to the introduction of this paper – see page 71.

On the infinity category of homotopy Leibniz algebras

The third article 'On the infinity category of homotopy Leibniz algebras' (homotopification, vertical categorification, higher category theory) is being published in the ArXiv preprint database and submitted for publication in a peer-reviewed international journal.

Various concepts of ∞-homotopies, as well as the relations between them (focussing on the Leibniz type) are discussed. In particular ∞-n-homotopies appear as the n-simplices of the nerve of a complete Lie ∞-algebra. In the nilpotent case, this nerve is known to be a Kan complex [Get09]. The authors argue that there is a quasi-category of ∞-algebras and show that for truncated ∞-algebras, i.e. categorified algebras, this ∞-categorical structure projects to a strict 2-categorical one. The paper contains a shortcut to $(\infty,1)$-categories, as well as a review of Getzler's proof of the Kan property. The latter is made concrete by applying it to the 2-term ∞-algebra case, thus recovering the concept of homotopy of [BC04], as well as the corresponding composition rule [SS07]. An answer to a question of [Sho08] about composition of ∞-homotopies of ∞-algebras is provided.

Again the reader is referred to the introduction of the paper for additional information – see page 98.

Appendix

In the appendix, we give proofs that were considered too technical to be part of a research paper.

2 Operadic approach to homotopy algebras

A good set-up for working with algebraic structures involves operads. This means that one considers all n-ary operations that can be obtained from some elementary 'building blocks', with all possible structures this collection naturally has. In the setting of homotopical algebra, identities between those operations only hold up to a hierarchy of coherent homotopies. Determining that hierarchy is an important problem, and there are several ways to approach it. One method that works in a large class of examples is described in the paper on Koszul duality for operads, by Ginzburg and Kapranov, which puts many disjoint ideas of homotopy theory in a uniform context.

We start our survey recalling the concepts of algebra and coalgebra. Then we introduce the bar and cobar constructions. We build the bar-cobar resolution of an algebra. Then we restrict ourselves to quadratic algebras, introduce the notions of Koszul (co)algebra and Koszul dual (co)algebra, and construct a 'smaller' resolution – the minimal model.

We give the classical and functorial definitions of an operad. The functorial one is similar to that of associative algebras. We adapt Koszul duality theory to this operadic framework. For each Koszul operad P, we define the P_∞ operad as the cobar construction of the Koszul dual cooperad of P. Finally we explain the Ginzburg-Kapranov theorem, which gives a compact characterization of P_∞-algebras by codifferentials on certain coalgebras.

2.1 Associative algebras and coalgebras

2.1.1 Notations and sign convention. Graded vector spaces.

Here we fix notations and give some basic definitions. We work over a field K of characteristic zero.

Definition 2.1.1. A vector space V is $\mathbb{Z}-$*graded (or just graded)* if there exists a collection of vector spaces $\{V_i\}, i \in \mathbb{Z}$, such that

$$V = \bigoplus_{i \in \mathbb{Z}} V_i.$$

Any element $a \in V_i$ is called a *homogeneous element of degree* i. The degree of this element is denoted by $\overline{a}$ or $|a|$.

The tensor product of two graded vector spaces has a natural grading. The degree of the tensor product of two homogeneous elements is equal to the sum of their degrees:

$$\overline{v \otimes w} = \overline{v} + \overline{w}, \quad v \in V_i, w \in W_j.$$

We dualize the graded vector space degree by degree:

$$V^* := \bigoplus_{i \in \mathbb{Z}} V_i^*.$$

To the elements of V_i^* we assign the degree $-i$. This kind of the dualization is not the 'honest' dualization $\mathrm{Hom}(V, K)$, but in the case if V is finite dimensional $V^* \simeq \mathrm{Hom}(V, K)$.

Definition 2.1.2. The *tensor module* over V is, by definition, a direct sum

$$T(V) := K \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

and the *reduced tensor module* over V is defined as follows:

$$\overline{T}(V) := V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

The vectors of the (reduced) tensor module which lie in the n-th tensor power of V: $v_1 \otimes ... \otimes v_n \in V^{\otimes n}$ are said to be of *weight* n. The vectors $v_1 \otimes ... \otimes v_n$ we usually denote by $v_1 ... v_n$, if it doesn't cause an ambiguity.

Definition 2.1.3. A linear map f between graded vector spaces V and W is called a *homogeneous map* of a degree $\overline{f} = k$, if it adds k to the degree of any homogeneous element: $fV_i \in W_{i+k}$.

For any homogeneous graded maps $f : V \to W$ and $f' : V' \to W'$ their tensor product $f \otimes f' : V \otimes V' \to W \otimes W'$ is defined in the following way:

$$(f \otimes f')(v \otimes v') = (-1)^{\overline{f'} \cdot \overline{v}} fv \otimes f'v'. \tag{2.1}$$

The transposed map $f^* : W^* \longrightarrow V^*$ defined as follows:

$$(f^* \alpha)v = (-1)^{\overline{f} \cdot \overline{\alpha}} \alpha(fv) \qquad v \in V, \alpha \in W^*. \tag{2.2}$$

Note that the dual spaces V^* and W^* are dualized, by definition, degree by degree, that is in general V^* and W^* are not isomorphic to $\mathrm{Hom}(V, K)$ and, respectively, $\mathrm{Hom}(W, K)$, nevertheless the map f^* is well defined. It follows from the fact that the map f has a certain degree k, and the map f^* sends all vectors from the vector space $(V_i)^*$ to the $(V_{i+k})^*$.
The way how the signs chosen in formulas (2.1) and (2.2) is called *the Koszul sign convention.* From this sign convention immediately follows that the transposition of composed maps

$$(fg)^* = (-1)^{\overline{f} \cdot \overline{g}} g^* f^* \tag{2.3}$$

and no sign appears in the transposition of tensor product:

$$(f \otimes g)^* = f^* \otimes g^*. \tag{2.4}$$

Note that in the last two formulas the argument's degree doesn't appear in the sign. That is one of the reasons that the Koszul sign convention helps to decrease the amount of signs in computations.

Chain and cochain complexes

Definition 2.1.4. A *chain complex* (V, d) is a graded vector space V together with a linear map d of degree -1 and satisfying $d^2 = 0$, called *a differential of the chain complex.*

$$... \xleftarrow{d} V_{-1} \xleftarrow{d} V_0 \xleftarrow{d} V_1 \xleftarrow{d} ...$$

Definition 2.1.5. A *cochain complex* (V, d) is a graded vector space V together with a linear map d of degree 1 and satisfying $d^2 = 0$, called *a differential of the cochain complex.*

$$... \xrightarrow{d} V^{-1} \xrightarrow{d} V^0 \xrightarrow{d} V^1 \xrightarrow{d} ...$$

Definition 2.1.6. A *morphism of (co)chain complexes* (V, d_V) and (W, d_W) is a linear map $f : V \to W$ of degree 0, which commutes with differentials, that is $d_W \circ f = f \circ d_V$.

Considering a chain complex (V, d), it is sometimes useful to denote the differential d, in a more explicit way, by $d_n : V_n \to V_{n-1}$. Note that $d^2 = 0$ explicitly reads as $d_n \circ d_{n+1}$, and thus $\mathrm{im} d_{n+1} \subset \mathrm{ker} d_n$. Elements of $\mathrm{ker} d_n$ are called *cycles* and elements of $\mathrm{im} d_{n+1}$ — *boundaries*. The *n-th homology group* is by definition

$$H_n := \mathrm{ker} d_n / \mathrm{im} d_{n+1}.$$

We denote $H_\bullet(V, d) := \bigoplus_{n \in \mathbb{Z}} H_n(V, D)$.

Similarly for the cochain complex (V, d), where we denote the differential, in more explicit way, by $d^n : V^n \to V^{n+1}$. The *n-th cohomology group* is by definition

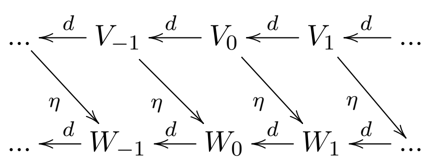
$$H^n := \mathrm{ker} d^n / \mathrm{im} d^{n-1}.$$

We denote $H^\bullet(V, d) := \bigoplus_{n \in \mathbb{Z}} H^n(V, D)$.

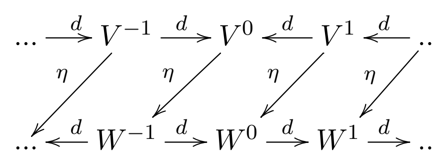
A (co)chain complex is called *acyclic* if its (co)homology is 0 everywhere. Note that a chain map $f : V \to W$ induces a linear map $f_\bullet$ in homology. If the map $f_\bullet$ is an isomorphism, we say that $f : V \xrightarrow{\sim} W$ is a *quasi-isomorphism.* Similarly one can define the notion of quasi-isomorphism for cochain maps.

Definition 2.1.7. A *(co)chain homotopy* between two (co)chain maps $f, g : (V, d_V) \to (W, d_W)$ is a map $\eta : V \to W$ of degree 1 (respectively -1), such that $\eta d + d\eta = g - f$,

chain complex



cochain complex



If two (co)chain maps are homotopic, the induced maps in (co)homology coincide.

2.1.2 Associative algebras

In this section we give definitions of associative algebras and coalgebras. The definitions are somehow dual to each other. It means that any associative algebra on a finite dimension vector space V gives rise to the coassociative coalgebra on V^* and vice versa. We will work in the graded framework.

Definition 2.1.8. *A graded associative algebra A over a field K* is a graded vector space A endowed with a K-linear 0-degree map

$$\mu : A \otimes A \to A,$$

which called a product (depending on a context the product μ denoted by a dot $(\cdot)$, star $(*)$, wedge $(\wedge)$,...), such that the associativity condition satisfies:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \Leftrightarrow \quad \mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu), \qquad a, b, c \in A.$$

Definition 2.1.9. A graded associative algebra A is called

- *unital*, if there exists an element $1_A \in A$, such that for any $a \in A$:

$$1_A \cdot a = a \cdot 1_A = a,$$

- *commutative*, if for any homogeneous elements $a, b \in A$:

$$a \cdot b = (-1)^{\bar{a} \cdot \bar{b}} b \cdot a,$$

- *anticommutative*, if for any homogeneous elements $a, b \in A$:

$$a \cdot b = -(-1)^{\bar{a} \cdot \bar{b}} b \cdot a.$$

One can show that if there exists the unit then it is unique. Indeed, let us assume that there exists another unit then $1'_A$, then

$$1'_A = 1'_A \cdot 1_A = 1_A.$$

Sometimes it is useful to express the associativity and unital properties in terms of commutative diagrams. Now we reformulate the axioms of associative algebras in this language. One of the advantages of the diagram approach that it doesn't refer to objects of an algebra.

associativity axiom:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}\otimes\mu} & A \otimes A \\ \Big\downarrow{\scriptstyle \mu\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \qquad \Leftrightarrow \qquad \mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu). \tag{2.5}$$

The existence of unit in the algebra A is equivalent to the existence of K-linear map $u : K \to A$, such that for any $k \in K$ and for any $a \in A$

$$ka = u(k) \cdot a = a \cdot u(k), \tag{2.6}$$

which can be expressed in the following commutative diagram:

unital axiom:

$$\begin{array}{ccccc} K \otimes A & \xrightarrow{u\otimes\mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id}\otimes u} & A \otimes K \\ & {\scriptstyle\cong}\searrow & \Big\downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle\cong} & \\ & & A & & \end{array} \qquad \Leftrightarrow \qquad \mu(u \otimes \mathrm{id}) = \mathrm{id} = \mu(\mathrm{id} \otimes u). \tag{2.7}$$

Definition 2.1.10. *An algebra homomorphism* between two graded associative algebras A and A' is a 0-degree linear map $F : A \to A'$ that respects the algebra structures:

$$F(a \cdot b) = F(a) \cdot F(b).$$

If algebras are unital then the map F is required to send the unit to unit:

$$F(1_A) = 1_{A'}.$$

Definition 2.1.11. Algebras A and A' are called *isomorphic* if there exists the homomorphism $F : A \to A'$ that is bijective. The map F is called an *algebra isomorphism.*

If $F : A \to A'$ is an algebra isomorphism then the map F^{-1} is also an algebra isomorphism. Indeed,

$$F^{-1}(a \cdot b) = F^{-1}(FF^{-1}a \cdot FF^{-1}b) = F^{-1}F(F^{-1}a \cdot F^{-1}b) = F^{-1}a \cdot F^{-1}b.$$

Definition 2.1.12. A unital algebra A is called *augmented* if there exists the subalgebra $\overline{A} \subset A$, such that the algebra A splits into two components

$$A = 1_A K \oplus \overline{A}.$$

The definition of the augmented algebra can be reformulated in the functorial spirit. One can show that the unital algebra is augmented if and only if there exists an algebra homomorphism $\varepsilon : A \to K$. This map is called the *augmentation map.*

Example 2.1.13. The *algebra of polynomials* $K[x_1, ..., x_n]$ is a commutative associative unital algebra.

Example 2.1.14. The *tensor algebra* over the tensor module

$$T(V) = K \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

which is given by the concatenation multiplication:

$$(v_1 \otimes ... \otimes v_p) \cdot (v_{p+1} \otimes ... \otimes v_{p+q}) = v_1 \otimes ... \otimes v_{p+q}.$$

The tensor algebra $(T(V), \cdot)$ is an associative unital algebra.

Example 2.1.15. The *reduced tensor algebra* over the reduced tensor module

$$\overline{T}(V) = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

with the concatenation product is an associative algebra.

Example 2.1.16. The *symmetric algebra* is by definition

$$S(V) := T(V)/I,$$

where the two-sided ideal I is generated by all differences of homogeneous elements' products: $v \otimes w - (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$ and the product descends from the tensor algebra's product. The reduced symmetric algebra is defined similarly:

$$\overline{S}(V) := \overline{T}(V)/I.$$

The symmetric algebra $S(V)$ is a graded commutative associative unital algebra. The reduced symmetric algebra $\overline{S}(V)$ is a graded commutative associative algebra. The (reduced) symmetric algebra can be decomposed in a direct sum of vector spaces:

$$S(V) = \bigoplus_{n=0}^{\infty} S^n(V), \quad \overline{S}(V) = \bigoplus_{n=1}^{\infty} S^n(V),$$

where $S^n(V)_{n\geqslant 0}$ is spanned by all elements of $S(V)$ of the weight n. If the vector space V is finite dimensional and has a dimension k then the symmetric algebra $S(V)$ is isomorphic to the algebra of polynomials $K[x_1, ..., x_k]$.

Definition 2.1.17. For any collection of homogeneous vectors $v_1, ..., v_n \in S(V)$ and for any permutation $\sigma \in S(n)$ the *Koszul sign* $\varepsilon(\sigma, v_1, ..., v_n)$ is defined by the commutativity relation

$$\varepsilon(\sigma, v_1, ..., v_n) v_{\sigma(1)} \cdot v_{\sigma(2)} \cdot ... \cdot v_{\sigma(n)} = v_1 \cdot v_2 \cdot ... \cdot v_n.$$

If it doesn't cause the confusions we will denote it just by $\varepsilon(\sigma)$.

Example 2.1.18. The exterior algebra is by definition

$$\Lambda(V) := T(V)/I,$$

where the two-sided ideal I is generated by all sums of homogeneous elements' products $v \otimes w +$ $+(-1)^{\overline{v}\cdot\overline{w}} w \otimes v$ and the product descends from the tensor algebra's product.
The reduced exterior algebra is defined similarly:

$$\overline{\Lambda}(V) := \overline{T}(V)/I.$$

The exterior algebra is anticommutative associative unital algebra. The reduced exterior algebra is anticommutative associative algebra. These algebras can be decomposed in a direct sum of vector spaces:

$$\Lambda(V) = \bigoplus_{n=0}^{\infty} \Lambda^n(V), \quad \overline{\Lambda}(V) = \bigoplus_{n=1}^{\infty} \Lambda^n(V),$$

where $\Lambda^n(V)$ is spanned by all elements of $S(V)$ of the weight n. The anticommutativity implies that for any homogeneous vectors $v_1, ..., v_n \in \Lambda(V)$ and for any permutation $\sigma \in S_n$ holds

$$v_1 \wedge v_2 \wedge ... \wedge v_n = \mathrm{sign}(\sigma) \cdot \varepsilon(\sigma, v_1, ..., v_n) v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge ... \wedge v_{\sigma(n)}.$$

In the case if the vector V is finite dimensional and possesses only 0-degree vectors then the exterior algebra is isomorphic to the algebra of linear forms over V^* with standard exterior product.

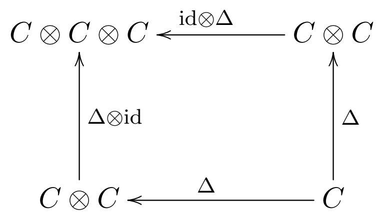
2.1.3 Coassociative coalgebras

The general idea to give definitions in the 'coalgebraic' world is to take the 'algebraic' definitions and revert all arrows in diagrams.

Definition 2.1.19. *A graded coassociative coalgebra C over a field K* is a graded vector space C endowed with a K-linear 0-degree map

$$\Delta : C \to C \otimes C,$$

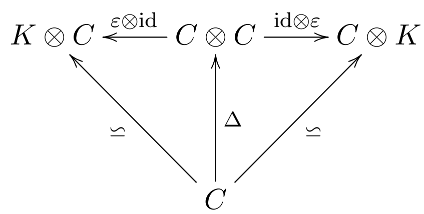
such that the it satisfies the coassociativity relation:



$$\Leftrightarrow \quad (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta. \tag{2.8}$$

The coalgebra C is called

- *counital* if there exists a map $\varepsilon : C \to K$ such that



$$\Leftrightarrow \quad (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta. \tag{2.9}$$

Remark 2.1.20. Consider the finite dimensional associative unital algebra (A, μ, u). The multiplication map μ and the unity map $u : K \to A$ satisfies respectively associativity and unitality conditions (2.5) and (2.7). Consider the transposed maps $\mu^* : A^* \to (A \otimes A)^*$ and $u^* : A^* \to K$. Since the vector space A is finite dimensional $(A \otimes A)^* \simeq A^* \otimes A^*$; the transposed associativity and unitality conditions will looks like:

$$(\mu^* \otimes \mathrm{id})\mu^* = (\mathrm{id} \otimes \mu^*)\mu^*,$$

$$(u^* \otimes \mathrm{id})\mu^* = \mathrm{id} = (\mathrm{id} \otimes u^*)\mu^*.$$

We see that if the associative unital algebra (A, μ, u) is finite dimensional then the datum (A^*, μ^*, u^*) form a coassociative counital coalgebra. Similarly any finite dimensional coalgebra (C, Δ, ε) gives rise to the finite dimensional algebra $(C^*, \Delta^*, \varepsilon^*)$.

Remark 2.1.21. In infinite-dimensional case the situation is more tricky. The spaces $(V \otimes W)^*$ and $V^* \otimes W^*$ are not isomorphic, but there exists an obvious embedding $V^* \otimes W^* \hookrightarrow (V \otimes W)^*$ defined by the formula $(\alpha \otimes \beta)(x \otimes y) \overset{def}{=} (-1)^{|x| \cdot |\beta|}\alpha(x)\beta(y)$. So for infinite-dimensional algebra (A, μ) the transposed map μ^* defines the coalgebra structure on A^* if and only if $\mathrm{Im}(\mu^*) \subset \mathrm{Im}(i)$:

$$\begin{array}{c} A \otimes A \xrightarrow{\ \mu\ } A \\ A^* \otimes A^* \overset{i}{\hookrightarrow} (A \otimes A)^* \xleftarrow{\ \mu^*\ } A^*. \end{array} \tag{2.10}$$

The situation for coalgebras is better. The transposed map Δ^* always defines a product on $C^* \otimes C^*$. Indeed,

$$C \xrightarrow{\Delta} C \otimes C \tag{2.11}$$
$$C^* \xleftarrow{\Delta^*} (C \otimes C)^* \xleftarrow{i} C^* \otimes C.^*$$

Definition 2.1.22. *A coalgebra homomorphism* between two graded associative coalgebras C and C' is a 0-degree linear map $F : C \to C'$ that respects the coalgebra structures:

$$\Delta F = (F \otimes F)\Delta. \tag{2.12}$$

If the coalgebras are counital then the map F is required to be compatible with counit maps:

$$\varepsilon F = \varepsilon'.$$

Example 2.1.23. If G is a finite group and $C(G)$ is a linear space of functions: $G \to K$. There is a natural coalgebra structure on the vector space $C(G)$. The coproduct

$$\Delta : C(G) \to C(G) \otimes C(G) \simeq C(G \times G)$$

is given by the following formula:

$$(\Delta f)(g_1, g_2) = f(g_1 \cdot g_2),\ \ g_1, g_2 \in G,$$

the counit map is given by

$$\varepsilon f = f(1_G).$$

The datum $(C(G), \Delta, \varepsilon)$ form a coassociative counital coalgebra

Example 2.1.24. *The tensor coalgebra* $T^c(V)$ as a vector space coincides with the tensor algebra $T(V)$. For any element $v_1...v_p \in V^{\otimes p} \subset T^c(V)$ the coproduct Δ acts in the following way:

$$\Delta(v_1...v_p) = 1_K \otimes v_1...v_p + \sum_{i=1}^{p-1} v_1...v_i \otimes v_{i+1}...v_p + v_1...v_p \otimes 1_K.$$

The coproduct Δ is called a *deconcatenation*. The counit map ε is the trivial projection $T^c(V) \to K$. the datum $(T^c(V), \Delta, \varepsilon)$ form a coassociative counital coalgebra. If the vector space V is finite dimensional then the tensor coalgebra $T^c(V)$ is dual to the tensor algebra, in the sense that the datum $(T(V^*), \Delta^*, \varepsilon^*)$ form a tensor algebra (with the concatenation product).

Similarly *the reduced tensor coalgebra* $\overline{T}^c(V)$ as a vector space coincides with the reduced tensor coalgebra $\overline{T}^c(V)$ and the coproduct $\overline{\Delta}$ is defined as follows:

$$\overline{\Delta}(v_1...v_p) = \sum_{i=1}^{p-1} v_1...v_i \otimes v_{i+1}...v_p.$$

Definition 2.1.25. A coalgebra C is called *coaugmented* if there exists the subcoalgebra $\overline{C} \subset C$, such that the coalgebra C splits into two components

$$C = 1_C K \oplus \overline{C}.$$

Similarly to the algebraic case this definition can be encrypted in the existence of the coalgebra homomorphism $u : K \to C$, which called the *coaugmentation map*.

Definition 2.1.26. To any finite set of natural numbers $i_1, ..., i_p$ one can correspond the set of $(i_1, ..., i_p)$-*unshuffled permutations* $Sh(i_1, ..., i_p)$, which is a set of all $(i_1 + ... + i_p)$-permutations, such that for any permutation $\sigma \in Sh(i_1, ..., i_p)$ the following conditions are required to satisfy:

$$\begin{aligned}
&\sigma(1) < ... < \sigma(i_1),\\
&\sigma(i_1 + 1) < ... < \sigma(i_1 + i_2),\\
&...\\
&\sigma(i_1 + ... + i_{p-1} + 1) < ... < \sigma(i_1 + ... + i_p).
\end{aligned}$$

If, in addition to this, the last elements in each group are ordered:

$$\sigma(i_1) < \sigma(i_1 + i_2) < ... < \sigma(i_1 + ... + i_p),$$

then we say that the permutation σ is an $(i_1, ..., i_p)$-*half-unshuffled permutation*. The set of such elements we denote by $Hsh(i_1, ..., i_p)$.

Example 2.1.27. *The symmetric coalgebra* $S^c(V)$ as a vector space coincides with the symmetric algebra $S(V)$. For any element $v_1...v_p \in S^c(V)$ the coproduct acts in the following way:

$$\begin{aligned}
\Delta(v_1...v_p) = 1_K \otimes (v_1...v_p) + \sum_{i=1}^{p-1} \sum_{\sigma \in Sh(i,p-i)} \varepsilon(\sigma, v_1, ..., v_p)(v_{\sigma(1)}...v_{\sigma(i)}) \otimes (v_{\sigma(i+1)}...v_{\sigma(p)})\\
+ (v_1...v_p) \otimes 1_K,
\end{aligned}$$

where $\varepsilon(\sigma, v_1, ..., v_p)$ is the Koszul sign. The counit map ε (don't confuse with the Koszul sign that is denoted by the same symbol) is the trivial projection $S^c(V) \to K$. The datum $(S^c(V), \Delta, \varepsilon)$ form a coassociative counital coalgebra. If the vector space V is finite dimensional then the symmetric coalgebra $S^c(V)$ is dual to the symmetric algebra, in the sense that the datum $(S^c(V^*), \Delta^*, \varepsilon^*)$ form the symmetric algebra.

2.1.4 Differential graded associative algebras and coalgebras

Definition 2.1.28. *A differential graded associative (DGA) algebra* (A, μ, d) is an associative algebra (A, μ) equipped with -1-degree map d, called *the differential*, which satisfies the following identities:

- $d(u \cdot v) = du \cdot v + (-1)^{\overline{u}} \cdot dv \quad \Leftrightarrow \quad d\mu = \mu(d \otimes \mathrm{id} + \mathrm{id} \otimes d)$ – d is a derivation,
- $d^2 = 0$.

Definition 2.1.29. *A differential graded associative (DGA) coalgebra* (C, Δ, D) is an coassociative coalgebra (C, Δ) equipped with -1-degree map D, called *the codifferential*, that satisfies the following identities:

- $\Delta D = (D \otimes \mathrm{id} + \mathrm{id} \otimes D)\Delta$ – D is a coderivation,
- $D^2 = 0$.

Definition 2.1.30. *A DGA algebra homomorphism* between DGA algebras A and A' is a graded algebra homomorphism $F : A \to A'$ that additionally respects the differential structure of algebras:

$$Fd - dF = 0.$$

Definition 2.1.31. *A DGA coalgebra homomorphism* between DGA coalgebras C and C' is a graded coalgebra homomorphism $F : C \to C'$ that additionally respects the codifferential structure of coalgebras:

$$FD - DF = 0.$$

Often (DGA) (co)algebras are imposed by additional non-negative grading, called *the weight*:

$$A := A^{(0)} \oplus A^{(1)} \oplus ...$$

If the weight is preserved by the (co)multiplication of the (co)algebra then we say that we deal with a *WGDA (weight-graded differential associative) (co)algebra.* For example, the tensor module $T(V)$ besides the homological grading that comes from the grading of V, carries the weight grading. The weight of each element in $V^{\otimes n} \subset T(V)$ is equal to n.

Definition 2.1.32. The WGDA (co)algebra A is called *connected* in the weight decomposition $A^{(0)} = K$.

Symmetric product

Consider maps $f, g : S(V) \to S(W)$. The vector spaces $S(V)$ and $S(W)$ may be endowed with symmetric product μ (symmetric algebra) as well as with coproduct Δ (symmetric coalgebra). The symmetric product $f \odot g : S(V) \to S(W)$ is defined as follows:

$$f \odot g := \mu \circ (f \otimes g) \circ \Delta.$$

For example, if the vector $v_1...v_p \in S^P(V)$ then

$$(f \odot g)(v_1...v_p) = \sum_{i+j=p} \sum_{\sigma \in Sh(i,j)} \varepsilon(\sigma) \cdot (-1)^{\overline{g} \cdot (\overline{v}_{\sigma(1)} + ... + \overline{v}_{\sigma(i)})} f(v_{\sigma(1)}...v_{\sigma(i)}) \cdot g(v_{\sigma(i+1)}...v_{\sigma(i+j)}).$$

The symmetric product is graded commutative:

$$f \odot g = (-1)^{\overline{f} \cdot \overline{g}} g \odot f.$$

The transposition

$$(f \odot g)^* = (\mu \circ (f \otimes g) \circ \Delta)^* = \Delta^* \circ (f^* \otimes g^*) \circ \mu^* = \mu \circ (f^* \otimes g^*) \circ \Delta = f^* \odot g^*.$$

Similarly, the symmetric product of n maps $f_1, f_2..., f_n$ is defined by

$$f_1 \odot ... \odot f_n := \mu^{n-1} \circ (f_1 \otimes ... \otimes f_n) \circ \Delta^{n-1},$$

where the n-th powers of multiplication and comultiplication are well defined because of their associativity. This also implies that the symmetric product is associative.

Suspension maps

Definition 2.1.33. For any graded vector space V *the shifted by* n *vector space* s^nV *(or* $V[n]$*)* is vector space, such that $V_i \simeq (s^nV)_{i+n}$.

In other words, there exists an isomorphism $s^n : V \to s^nV$. The linear map s^n does nothing but increases degrees of vectors by n, so it has a degree n. The map s is called a *suspension map* and the s^{-1} is a *desuspension map*.
The dual of shifted vector space $(s^nV)^* \simeq s^{-n}V^*$.

For the n-th tensor power of (de)suspension maps: $V^{\otimes n} \underset{(s^{-1})^{\otimes n}}{\overset{s^{\otimes n}}{\rightleftarrows}} (sV)^{\otimes n}$, according to the Koszul sign rule, their composition satisfy the following relations:

$$s^{\otimes n} \circ (s^{-1})^{\otimes n} = (s^{-1})^{\otimes n} \circ s^{\otimes n} = (-1)^{\frac{n(n-1)}{2}} \mathrm{id}^{\otimes n}. \tag{2.13}$$

For any map $f : (sV)^{\otimes n} \to (sW)^{\otimes k}$ one can define the suspended map $f^{\text{susp}} : V^{\otimes n} \to W^{\otimes k}$ via the following diagram:

$$\begin{array}{ccc} V \otimes ... \otimes V & \xrightarrow{f^{\text{susp}}} & W \otimes ... \otimes W \\ {\scriptstyle s^{\otimes n}}\downarrow & & \uparrow{\scriptstyle (s^{-1})^{\otimes k}} \\ sV \otimes ... \otimes sV & \xrightarrow{f} & sW \otimes ... \otimes sW \end{array} \quad \Leftrightarrow \quad f^{\text{susp}} := (s^{-1})^{\otimes k} \circ f \circ (s)^{\otimes n}.$$

We see that $\overline{f^{\text{susp}}} = \overline{f} + n - k$.

One can show that for the permutation map $\sigma_n : V^{\otimes n} \to V^{\otimes n}$ the following identity holds:

$$\sigma_n^{\text{susp}} = (-1)^{\frac{n(n-1)}{2}} \mathrm{sign}(\sigma_n) \cdot \sigma_n. \tag{2.14}$$

Proposition 2.1.34. *For any graded vector space* V *the exterior algebra* $\Lambda(V)$ *is isomorphic to* $\bigoplus_{n=0}^{\infty} (s^{-1})^{\otimes n} S^n(sV)$.

Proof. We should prove that the map $(s^{-1})^{\otimes n} : S^n(sV) \to \Lambda(V)$ is well defined on the $S^n(sV) = (sV)^{\otimes n}/I$, that is it preserves the equivalence classes. We check this for the case when $n = 2$. Take the representative element $sv_1 \otimes sv_2$ in the equivalence class $sv_1 \cdot sv_2 \in S^2(sV)$:

$$[(s^{-1} \otimes s^{-1})(sv_1 \otimes sv_2)] = [(-1)^{v_1+1} v_1 \otimes v_2] = (-1)^{v_1+1} v_1 \wedge v_2.$$

Take the other element $(-1)^{(v_1+1)(v_2+1)}(sv_2 \otimes sv_1)$ which belongs to the same class $sv_1 \cdot sv_2$. The map $(s^{-1} \otimes s^{-1})$ sends this element to the same equivalence class $(-1)^{v_1+1}[v_1 \wedge v_2]$. Indeed,

$$[(s^{-1} \otimes s^{-1})(-1)^{(v_1+1)(v_2+1)}(sv_2 \otimes sv_1)] = [(-1)^{v_1v_2+1} \cdot (-1)^{v_1+1} v_2 \otimes v_1] = (-1)^{v_1+1} v_1 \wedge v_2.$$

Similarly one can show that the map $(s^{-1})^{\otimes n}$ is well defined for any n. Thus we prove the space isomorphism. $\square$

For any map $f : S^n(sV) \to S^k(sW)$ one can define the suspended map $f^{\text{susp}} : \Lambda^n(V) \to \Lambda^k(W)$ via the following diagram:

$$\begin{array}{ccc} \Lambda^n(V) & \xrightarrow{f^{\text{susp}}} & \Lambda^k(W) \\ {\scriptstyle s^{\otimes n}}\downarrow & & \uparrow{\scriptstyle (s^{-1})^{\otimes k}} \\ S^n(sV) & \xrightarrow{f} & S^k(sW) \end{array} \qquad \Leftrightarrow \qquad f^{\text{susp}} \overset{def}{=} (s^{-1})^{\otimes k} \circ f \circ (s)^{\otimes n}.$$

We see that $\overline{f^{\text{susp}}} = \overline{f} + n - k$.

2.2 Koszul theory on associative algebras and coalgebras

This section deals with twisting and Koszul morphisms for associative algebras and coalgebras. Moreover, we take a special interest in the bar and cobar construction, which will finally provide a model (given by the bar-cobar resolution) of the considered differential graded associative algebra. Then we restrict ourselves to specific type of (co)algebras, namely 'quadratic Koszul (co)algebras' and construct for them a 'smaller' model.

2.2.1 Twisting morphisms and twisted tensor complexes

In this section A is a augmented unital DGA algebra and C is a counital coaugmented DGA coalgebra.
The *convolution algebra* in the space of linear maps $\text{Hom}(C, A)$ is given by the product

$$f \star g = \mu \circ (f \otimes g) \circ \Delta.$$

The unit of the product $\star$ is given by $u \circ \varepsilon \in \text{Hom}(C, A)$. The associativity of the product follows from the associativity of the product and coproduct on A and C.

Proposition 2.2.1. *The map*

$$\partial f = d_A \circ f - (-1)^{\overline{f}} f \circ d_C$$

is a differential on the convolution algebra $(\text{Hom}(C, A), \star)$.

Proof. Take any homogeneous elements $f, g \in \text{Hom}(C, A)$

$$\begin{aligned} \partial(f \star g) &= d_A \circ (f \star g) - (-1)^{\overline{f}+\overline{g}} (f \star g) \circ d_C \\ &= d_A \circ \mu \circ (f \otimes g) \circ \Delta - (-1)^{\overline{f}+\overline{g}} \mu \circ (f \otimes g) \circ \Delta \circ d_C \\ &= \mu \circ (d_A f - (-1)^{\overline{f}} f d_C \otimes g) \circ \Delta + (-1)^{\overline{f}} \mu \circ (f \otimes d_A g - (-1)^{\overline{g}} g d_C) \circ \Delta \circ d_C \\ &= \partial f \star g + (-1)^{\overline{f}} f \star \partial g. \end{aligned}$$

This shows that the operation ∂ is a derivation of degree -1 on the convolution algebra. The derivation ∂ also satisfies the condition $\partial^2 = 0$, indeed

$$\begin{aligned} \partial(\partial f) &= \partial(d_A f - (-1)^{\overline{f}} f d_C) \\ &= d_A^2 f - (-1)^{\overline{f}+1} d_A f d_C - (-1)^{\overline{f}} d_A f d_C - f d_C^2 = 0. \end{aligned}$$

So the derivation ∂ is a differential. $\square$

Definition 2.2.2. The *twisting morphism* $\alpha \in \mathrm{Tw}(C, A)$ is a morphism $\alpha \in \mathrm{Hom}_K(C, A)$ of degree -1, that satisfies the *Maurer-Cartan equation*

$$\partial\alpha + \alpha \star \alpha = 0, \tag{2.15}$$

and which is null when composed with the augmentation of A and also when composed with the coaugmentation of C.

The last condition is of technical purpose and can be formulated as

$$\alpha \circ u = 0 \Leftrightarrow \alpha(1_C) = 0 \quad \text{and} \quad \varepsilon \circ \alpha = 0 \Leftrightarrow \alpha(\overline{C}) \subset \overline{A}.$$

where $u : K \to C$ denotes the coaugmentation map and $\varepsilon : A \to K$ the augmentation map.

Consider the tensor complex $(C \otimes A, d_{C\otimes A})$, where $d_{C\otimes A} = d_C \otimes \mathrm{id} + \mathrm{id} \otimes d_A$. Moreover, consider a morphism $\alpha \in \mathrm{Hom}_K(C, A)$ and $\overline{d}_\alpha : C \otimes A \to C \otimes A$ defined by the following diagram:

$$C \otimes A \xrightarrow{\Delta\otimes\mathrm{id}} C \otimes C \otimes A \xrightarrow{\mathrm{id}\otimes\alpha\otimes\mathrm{id}} C \otimes A \otimes A \xrightarrow{\mathrm{id}\otimes\mu} C \otimes A.$$

Proposition 2.2.3. *For any map $\alpha : C \to A$ of degree -1 the condition $d_\alpha^2 = 0$ (where $d_\alpha = d_{C\otimes A} + \overline{d}_\alpha$) holds true if and only if the map α satisfies the Maurer-Cartan equation (2.15).*

Proof.

$$d_\alpha^2 = (d_{C\otimes A} + \overline{d}_\alpha)^2 = d_{C\otimes A}^2 + d_\alpha^2 + d_{C\otimes A} \circ \overline{d}_\alpha + \overline{d}_\alpha \circ d_{C\otimes A},$$

where $d_{C\otimes A}^2 = 0$, because $d_{C\otimes A}$ is a differential of the complex $C \otimes A$.
The next term $\overline{d}_\alpha^2$ is equal to $\overline{d}_{\alpha\star\alpha}$, indeed

$$\begin{aligned}
\overline{d}_\alpha^2 &= (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \\
&= (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \mu) \circ (\Delta \otimes \mathrm{id}^{\otimes 2}) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \\
&= (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha \otimes \alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \\
&= (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \alpha \star \alpha \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) = \overline{d}_{\alpha\star\alpha}.
\end{aligned}$$

Finally for the last two terms $d_{C\otimes A} \circ \overline{d}_\alpha + \overline{d}_\alpha \circ d_{C\otimes A} = \overline{d}_{\partial(\alpha)}$, indeed

$$\begin{aligned}
&d_{C\otimes A} \circ \overline{d}_\alpha + \overline{d}_\alpha \circ d_{C\otimes A} = \\
&= (d_C \otimes \mathrm{id} + \mathrm{id} \otimes d_A) \circ (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) + \\
&+ (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ (d_C \otimes \mathrm{id} + \mathrm{id} \otimes d_A) \\
&= (\mathrm{id} \otimes \mu) \circ (d_C \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes d_A \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes d_A) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) + \\
&+ (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (d_C \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes d_C \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes d_A) \circ (\Delta \otimes \mathrm{id}) \\
&= (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes d_A\alpha + \alpha d_C \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) = \overline{d}_{\partial(\alpha)}.
\end{aligned}$$

We see that

$$d_\alpha^2 = \overline{d}_{\alpha\star\alpha+\partial(\alpha)}.$$

If the map α satisfies the Maurer-Cartan equation (2.15) then $\overline{d}_{\alpha\star\alpha+\partial(\alpha)} = 0$.
The converse statement follows from the fact that for any map $f : C \to A$ the restriction of the map $\overline{d}_f$ on $C \otimes K \to K \otimes A$ is equal to f, indeed

$$\begin{aligned}(\varepsilon \otimes \mathrm{id}) \circ \overline{d}_f \circ (\mathrm{id} \otimes u) &= (\varepsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes f \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes u) \\ &= \underbrace{(\mathrm{id} \otimes \mu) \circ (\mathrm{id}^{\otimes 2} \otimes u)}_{\mathrm{id}\otimes\simeq} \circ (\mathrm{id} \otimes f \otimes \mathrm{id}) \circ \underbrace{(\varepsilon \otimes \mathrm{id}^{\otimes 2}) \circ (\Delta \otimes \mathrm{id})}_{\simeq\otimes\mathrm{id}} \\ &= f.\end{aligned}$$

□

Defined in the preceding proposition differential d_α is called the *perturbation of the differential* $d_{C\otimes A}$, and the complex $C \otimes_\alpha A := (C \otimes A, d_\alpha)$ is called the *twisted tensor complex*.

Lemma 2.2.4 (Comparison lemma for twisted tensor complexes)**.** *Let $g : A \to A'$ be a morphism of weight-DGA connected algebras and $f : C \to C'$ be a morphism of weight-DGA connected coalgebras. Let $\alpha : C \to A$ and $\alpha' \to C' \to A'$ be two twisting morphisms, such that f and g are compatible with α and α', that is the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C' & \xrightarrow{\alpha'} & A' \end{array}$$

If two morphisms among f, g and $f \otimes g : C \otimes_\alpha A \to C' \otimes_{\alpha'} A'$ are quasi-isomorphisms, then so is the third one.

2.2.2 Bar and cobar complexes

Consider an arbitrary DGA coalgebra C with codifferential $D : C \to C$ and comultiplication $\Delta : C \to C \otimes C$. The idea behind the cobar construction is to encode the information about the codifferential D and the comultiplication Δ in a derivation of the tensor algebra $T(C)$ (up to suspensions). A derivation on $T(C)$ is characterized by its restriction on C. Up to suspension we define it to be $\Delta + D : C \to T(C)$. The coassociativity of Δ and the fact that D is a codifferential imply that the constructed derivation is a differential.
Let us be more precise, consider an augmented conilpotent DGA coalgebra $(C = 1_C K \oplus \overline{C}, \Delta, d_C)$ and the tensor algebra $T(s^{-1}\overline{C})$. Any derivation d on $T(s^{-1}\overline{C})$ is given by its restriction on $s^{-1}\overline{C}$.

$$d|_{s^{-1}\overline{C}} = d_1 + d_2 + ..., \text{ where } d_k : s^{-1}\overline{C} \in \left(s^{-1}\overline{C}\right)^{\otimes k}.$$

For any element $v_1...v_n \in T(s^{-1}\overline{C})$ its derivation

$$d(v_1...v_n) = \sum_{i=1}^{n} (\mathrm{id}^{(i-1)} \otimes d \otimes \mathrm{id}^{(n-i)})(v_1...v_n) = \sum_{i=1}^{n}\sum_{k=1}^{\infty} (\mathrm{id}^{(i-1)} \otimes d_k \otimes \mathrm{id}^{(n-i)})(v_1...v_n).$$

Definition 2.2.5. The *cobar complex* $\Omega C = (T(s^{-1}\overline{C}), d_{\Omega C})$ is defined by the -1 degree derivation $d_{\Omega C}$, which is given by restrictions

$$\begin{aligned} d_1 &= -s^{-1} d_C s \quad (s^{-1}\overline{C} \to s^{-1}\overline{C}), \\ d_2 &= -(s^{-1} \otimes s^{-1})\Delta s \quad (s^{-1}\overline{C} \to \left(s^{-1}\overline{C}\right)^{\otimes 2}), \\ d_k &= 0, \text{ for other } k. \end{aligned}$$

Proposition 2.2.6. *Defined above derivation $d_{\Omega C}$ is a differential, that is $d^2_{\Omega C} = 0$.*

Proof. Since $d^2_{\Omega C}$ is a derivation, it is determined by its action on generators:

$d^2_{\Omega C}(v) = d_{\Omega C} \circ (d_1 + d_2)(v) = (d_1^2 + (\mathrm{id} \otimes d_1)d_2 + (d_1 \otimes \mathrm{id})d_2 + d_2 d_1 + (d_2 \otimes \mathrm{id})d_2 + (\mathrm{id} \otimes d_2)d_2)(v)$.

The right-hand side of the last equation splits into three parts:

1. $d_1^2 = (s^{-1}d_C s)^2 = s^{-1}d_C^2 s = 0$ (d_C - is a differential),
2. $(\mathrm{id} \otimes d_1)d_2 + (d_1 \otimes \mathrm{id})d_2 + d_2 d_1 = (s^{-1} \otimes s^{-1})(-(\mathrm{id} \otimes d_C + d_C \otimes \mathrm{id})\Delta + \Delta d_C)s = 0$ (d_C is a derivation for the coproduct Δ),
3. $(d_2 \otimes \mathrm{id})d_2 + (\mathrm{id} \otimes d_2)d_2 = (s^{-1} \otimes s^{-1} \otimes s^{-1})((\Delta \otimes \mathrm{id})\Delta - (\mathrm{id} \otimes \Delta)\Delta)s = 0$ (coassociativity of the coproduct Δ).

Finally we see that the $d^2_{\Omega C} = 0$, so the $d_{\Omega C}$ is a differential. □

Similarly, in the dual case consider an augmented DGA algebra $(A = 1_A K \oplus \overline{A}, \mu, d_A)$ and the tensor coalgebra $T^c(s\overline{A})$. Any coderivation D on $T^c(s\overline{A})$ is given by its corestriction's $D_n : (s\overline{A})^{\otimes n} \to s\overline{A}$:

$$D(v_1 ... v_n) = \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} (\mathrm{id}^{(i-1)} \otimes D_k \otimes \mathrm{id}^{(n-i-k+1)})(v_1 ... v_n).$$

Definition 2.2.7. The *bar complex* $\mathrm{B}A = (T^c(s\overline{A}), D_{\mathrm{B}A})$ is defined by the -1 degree coderivation $D_{\mathrm{B}A}$, which is given by corestrictions

$$\begin{aligned} &D_1 = -s d_A s^{-1} \quad (s\overline{A} \to s\overline{A}),\\ &D_2 = -s\mu(s^{-1} \otimes s^{-1}) \quad ((s\overline{A})^{\otimes 2} \to s\overline{A}),\\ &D_k = 0, \text{ for other } k. \end{aligned}$$

Proposition 2.2.8. *Defined above coderivation $d_{\mathrm{B}A}$ is a codifferential, that is $d^2_{\mathrm{B}A}$.*

Proof. The proof is similar to the one for cobar complex. □

Proposition 2.2.9. *There is an isomorphism of spaces:*

$$\mathrm{Hom}_{DGAA}(\Omega C, A) \simeq \mathrm{T}w(C, A) \simeq \mathrm{Hom}_{DGAC}(C, BA).$$

Proof. We will prove only the first isomorphism, the proof for the second one will be similar. Any DGA algebra homomorphism $F : \Omega(C) \simeq T(s^{-1}\overline{C}) \to A$ is given by its restriction $f : s^{-1}\overline{C} \to A$. The condition $Fd - dF = 0$ on the generators reads as

$$\begin{aligned} &F(d_1 + d_2) - d_A f = 0,\\ &- f s^{-1} d_C s - F(s^{-1} \otimes s^{-1}) d_c s - d_A f s^{-1} s = 0,\\ &f s^{-1} d_C + \mu(f s^{-1} \otimes f s^{-1}) d_c + d f_A s^{-1} = 0,\\ &\partial\alpha + \alpha \star \alpha = 0, \end{aligned}$$

where the restriction of the map $\alpha : C \to A$ on the $\overline{C}$ is $\alpha|_{\overline{C}} := fs^{-1}$ and $\alpha(1_C) := 0$. Note that α satisfies the Maurer-Cartan equation. Since the F is the homomorphism of augmented algebras, that is it respects the augmentation structure, $\alpha(\overline{C}) \subset \overline{A}$. We see that the map $\alpha \in \mathrm{Tw}(C, A)$. □

Remark 2.2.10. Note that the spaces $\mathrm{Hom}_{DGAA}(\Omega C, A)$ and $\mathrm{Hom}_{DGAC}(C, BA)$ are the spaces of homomorphisms of the (co)augmented (co)algebras, which means, by definition, that they respect the (co)augmentation structures. This corresponds to the fact that twisting morphisms have to vanish on units and counits.

2.2.3 Koszul morphisms and bar-cobar resolution

Definition 2.2.11. A twisting morphism $\alpha \in \mathrm{Tw}(C, A)$ is called a *Koszul morphism*, if the twisted tensor complex $C \otimes_\alpha A$ is acyclic. The set of Koszul morphisms from C to A is denoted by $\mathrm{Kos}(C, A)$

Theorem 2.2.12 (Fundamental theorem of twisting morphisms)**.** *Let A be a connected WGDA algebra and C be a connected WGDA conilpotent coalgebra. For any twisting morphism $\alpha \in \mathrm{Tw}(C, A)$ and constructed from it homomorphisms $f_\alpha \in \mathrm{Hom}_{DGAA}(\Omega C, A)$ and $g_\alpha \in \mathrm{Hom}_{DGAC}(C, BA)$ the following propositions are equivalent:*

1. *$\alpha \in \mathrm{Kos}(C, A)$, that is the twisted complex $C \otimes_\alpha A$ is acyclic,*

2. *f_α is a quasi-isomorphism,*

3. *g_α is a quasi-isomorphism.*

$$\mathrm{QIso}_{DGAA}(\Omega C, A) \simeq \mathrm{Kos}(C, A) \simeq \mathrm{QIso}_{DGAC}(C, BA). \tag{2.16}$$

Proof. We will prove the equivalence of the first and the second propositions. To the identity map $\mathrm{id} \in Hom_{DGAC}(\Omega C, \Omega C)$ one can correspond the twisting morphism $i \in \mathrm{Tw}(C, \Omega C)$ due to the Proposition 2.2.9. Since the twisted complex $C \otimes_i \Omega C$ is acyclic, the map $\mathrm{id} \otimes f_\alpha$ is a quasi-isomorphism. Obviously the morphism $\mathrm{id} : C \to C$ is a quasi-isomorphism as well. We apply the comparison lemma for the commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\ i\ } & \Omega C \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ C & \xrightarrow{\ \alpha\ } & A \end{array}$$

and get that f_α is a quasi-isomorphism. □

Take BA for the coalgebra C, then one can correspond to the identity map $\mathrm{id} \in \mathrm{QIso}_{DGAC}(BA, BA)$ the map $\varepsilon \in \mathrm{QIso}_{DGAC}(\Omega BA, A)$ (due to the isomorphisms (2.16)). We say that ε is a *bar-cobar resolution* and that ΩBA is a *model* of A. The model ΩBA generally is too 'big'. In the next section we will study when it is possible to replace it by a smaller one.

2.2.4 Quadratic algebras and coalgebras. Koszul dual

Definition 2.2.13. *Quadratic data* (V, R) consists of a graded vector space V and a graded vector subspace $R \subset V^{\otimes 2}$

Definition 2.2.14. The *quadratic algebra* $A(V, R)$ associated to the quadratic data (V, R) is the quotient algebra $T(V)/(R)$, where (R) denotes the two-sided ideal generated by R

Note that

$$(R) = \bigoplus_{n \geqslant 2} \sum_{i+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}$$

and that

$$A(V, R) = K \oplus V \oplus V^{\otimes 2}/R \oplus ... \oplus \left(V^{\otimes n} \Big/ \sum_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus ... =: \bigoplus_{n \geqslant 0} A^{(n)}(V, R).$$

Definition 2.2.15. The *quadratic coalgebra* $C(V, R)$ associated to the quadratic data (V, R) is the coalgebra of the tensor algebra $T^c(V)$, defined on the vector space

$$C(V, R) = K \oplus V \oplus R \oplus (V \otimes R \cap R \otimes V) \oplus ... \oplus \left(\bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus ... =: \bigoplus_{n \geqslant 0} C^{(n)}(V, R).$$

The definition of quadratic coalgebra is dual to the definition of quadratic algebras, in the sense that if the vector space V is finite dimensional then the coalgebra $A^*(V, R) \simeq C(V^*, R^{\perp})$. Indeed,

$$\begin{aligned} A^{*(n)}(V, R) &= \left(V^{\otimes n} \Big/ \sum_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right)^* \\ &= \left(\sum_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right)^{\perp} = V^{*\otimes i} \otimes R^{\perp} \otimes V^{*\otimes j} = C^{(n)}(V^*, R^{\perp}). \end{aligned}$$

Definition 2.2.16. The *Koszul dual coalgebra* of a quadratic algebra $A(V, R)$ is

$$A^{¡}(V, R) := C(sV, s^2 R)$$

Definition 2.2.17. The *Koszul dual algebra* of a quadratic algebra $A(V, R)$ is

$$A^{!(n)} := s^n (A^{¡*})^{(n)}$$

Definition 2.2.18. The *Koszul dual algebra* of a quadratic coalgebra $C(V, R)$ is

$$C^{¡}(V, R) := A(s^{-1}V, s^{-2}R)$$

Definition 2.2.19. The *Koszul dual coalgebra* of a quadratic coalgebra $C(V,R)$ is

$$C^{!(n)} := s^{-n}(C^{¡*})^{(n)}$$

One can verify that $(A^{¡})^{¡} = A$ and $(C^{¡})^{¡} = C$ and in the finite dimensional case $\left(A^{!}\right)^{!} = A$, $\left(C^{!}\right)^{!} = C$ and $A^{!} = A(V^*, R^{\perp})$.

Example 2.2.20. If V is a finite dimensional graded vector space and the vector space $R = \langle v\otimes w - (-1)^{|v||w|} w\otimes v\rangle$, then

1. $A(V,R) = T(V)/(R) = S(V)$,
2. $A^{¡} = C(sV, s^2R) = \Lambda^c(sV)$,
3. $A^{!} = \Lambda(V^*)$,
4. $C(V,R) = \Lambda^c(V)$,
5. $C^{¡} = A(s^{-1}V, s^{-2}R) = S(s^{-1}V)$,
6. $C^{!} = S^c(V^*)$.

Example 2.2.21. If V is a finite dimensional graded vector space and the vector space $R = \langle v\otimes w + (-1)^{|v||w|} w\otimes v\rangle$, then

1. $A(V,R) = T(V)/(R) = \Lambda(V)$,
2. $A^{¡} = C(sV, s^2R) = S^c(sV)$,
3. $A^{!} = S(V^*)$,
4. $C(V,R) = S^c(V)$,
5. $C^{¡} = A(s^{-1}V, s^{-2}R) = \Lambda(s^{-1}V)$,
6. $C^{!} = \Lambda^c(V^*)$.

2.2.5 Koszul algebras

We now replace for quadratic algebras, under certain conditions, the 'big' resolution $\Omega BA \xrightarrow{\sim} A$ by a smaller one, namely $\Omega A^{¡} \xrightarrow{\sim} A$, which is its minimal model.
We define the twisting morphism by

$$\kappa :\ A^{¡} = C(sV, s^2R) \xrightarrow{p} sV \xrightarrow{s^{-1}} V \xrightarrow{i} A(V,R) = A.$$

Since the DGA coalgebra $A^{¡}$ and the DGA algebra A have zero differential, the Maurer-Cartan equation will simplify: $\kappa\star\kappa = 0$. To verify that the defined above map κ is a twisting morphism, it is sufficient to check that $\kappa\star\kappa$ is zero on $s^2R \subset sV^{\otimes 2}$:

$$[(\kappa\star\kappa)sr_1\otimes sr_2] = [\kappa(sr_1)\kappa(sr_2)] = [r_1r_2] = 0.$$

All other term automatically goes to the zero, since the map κ is not zero only on sV.

Definition 2.2.22. The twisted complex $A^{¡} \otimes_\kappa A$ is called *the Koszul complex.* If the Koszul complex is acyclic then the algebra A is called a *Koszul algebra.*

If the algebra A is Koszul then due to the fundamental theorem of twisting morphisms (Theorem 2.2.12) the projection $p : \Omega A^{¡} \to A$ is a quasi-isomorphism, that is the $\Omega A^{¡}$ is a resolution of A.

2.3 Operads

In this section we give the classical and functorial definitions of operad. The classical definition corresponds to our intuition. The functorial one is defined in a similar way as the associative algebra is defined. This helps to translate the Koszul duality theory for associative algebras to the operadic framework.

2.3.1 Classical definition of operad

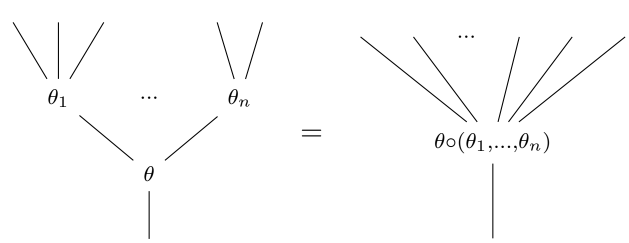
In this section we work in the non graded framework. The action of the symmetric group S_n is defined in the following way:

- the left S_n action:
$$\sigma(v_1 \otimes ... \otimes v_n) = v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(n)},$$
- the right S_n action:
$$(v_1 \otimes ... \otimes v_n)\sigma = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}.$$

Definition 2.3.1. An *S-module* P is a sequence $\{P(n)\}, n \in \mathbb{N}$ of vector spaces endowed with right S_n-module structures.

Definition 2.3.2. *A symmetric operad* consists of the following:

- S-module $P = \{P(n)\}$, whose elements are called *n-ary operations*,
- *an identity element* $\mathbf{1} \in P(1)$,
- for all positive integers $n, k_1, ..., k_n$ a composition function
$$\circ : P(n) \otimes P(k_1) \otimes ... \otimes P(k_n) \to P(k_1 + ... + k_n)$$
$$(\theta, \theta_1, ...\theta_n) \to \theta \circ (\theta_1, ..., \theta_n),$$
pictorially this composition looks as follows:



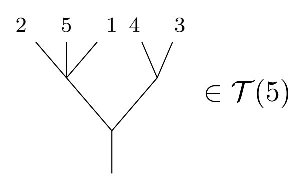
satisfying the following coherence axioms:

1. identity: $\theta \circ (\mathbf{1}, ..., \mathbf{1}) = \theta = \mathbf{1} \circ \theta$,
2. associativity:
$$\theta \circ \Big(\theta_1 \circ (\theta_{1,1}, ..., \theta_{1,k_1}), ..., \theta_n \circ (\theta_{n,1}, ..., \theta_{n,k_n})\Big) = \Big(\theta \circ (\theta_1, ..., \theta_n)\Big) \circ (\theta_{1,1}, ..., \theta_{1,k_1}, ..., \theta_{n,1}, ..., \theta_{n,k_n}),$$
3. equivariance: given permutations $\sigma \in S_n$ and $\pi_i \in S_{k_i}$
$$(\theta\sigma) \circ (\theta_1, ..., \theta_n) = (\theta \circ (\theta_{\sigma^{-1}(1)}, ..., \theta_{\sigma^{-1}(n)}))\overline{\sigma}$$
(where $\overline{\sigma} \in S_{k_1+...+k_n}$ on RHS is permuting n groups in the same way as $\sigma \in S_n$ permutes n elements),
$$\theta \circ (\theta_1\pi_1, ..., \theta_n\pi_n) = (\theta \circ (\theta_1, ..., \theta_n))(\pi_1, ..., \pi_n).$$

Example 2.3.3. The *endomorphism operad* $\mathcal{E}nd(V)$ over a vector space V is made up by the vector spaces $\mathcal{E}nd(V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$ the usual composition and the identity map id_V. The right S_n module structure on $\mathrm{Hom}(V^{\otimes n}, V)$ is given by
$$(f\sigma)(v_1 \otimes ... \otimes v_n) := f(\sigma(v_1 \otimes ... \otimes v_n)) = f(v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(n)})$$

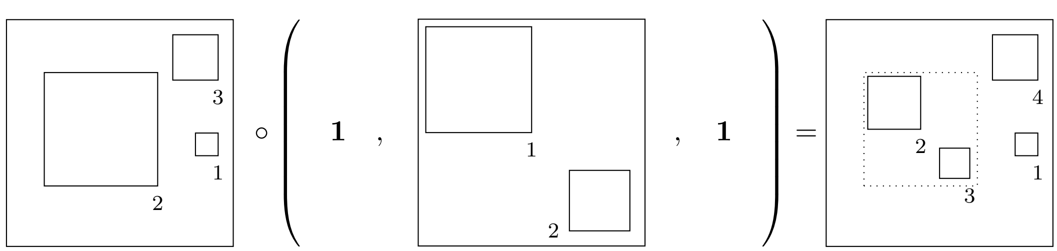
Remark 2.3.4. Note that the associativity and equivariance axioms do not mean that the n-ary operations that are encoded in the operad are associative and symmetric (whatever it means). You might see it in the previous example for the operad $\mathcal{E}nd(V)$, where these axioms hold tautologically. The existence of these axioms in the definition of the operad is the shadow of the fact, that we want to have a well defined notion of a representation of any operad P in $\mathcal{E}nd(V)(n)$ (see the next section the precise definition of representation).

Example 2.3.5. The operad of *labeled planar trees* $\mathcal{T}$. The vector space $\mathcal{T}(n)$ is spanned by planar trees that have a root edge and n leaves labeled by integers 1 through n. For instance:



The composition is given by grafting on the leaves. Where the order of grafting is defined by the labels of the leaves.

Example 2.3.6. The *little n-cubes operad* C_n. The elements of the vector space $C_n(j)$ are the ordered collections of j n-cubes linearly embedded in the standard n-dimensional unit cube I^n with disjoint interiors and axes parallel to those of I^n. The compositions are given as indicated here:



2.3.2 Morphisms and representations of operads

Operads are important through their representations. In order to define representations of operads, we first have to define morphisms of operads.

Definition 2.3.7. An *operad morphism* $\varphi : P \to Q$ consists of the sequence of linear maps $\varphi_n : P(n) \to Q(n)$, which for all operations $\theta \in P(n), \theta_i \in P(k_i)$

- respect the composition:

$$\varphi_n(\theta \circ_P (\theta_1, ..., \theta_n)) = \varphi_n(\theta) \circ_Q (\varphi_{k_1}(\theta_1), ..., \varphi_{k_n}(\theta_n)),$$

- respect the unit:

$$\varphi_1(\mathbf{1}_P) = \mathbf{1}_Q,$$

- for any permutation $\sigma \in S(n)$

$$\varphi_n(\theta\sigma) = \varphi_n(\theta)\sigma.$$

Definition 2.3.8. A *representation of operad P on a vector space V* is an operad morphism $\rho : P \to \mathcal{E}nd(V)$

Remark 2.3.9. Any representation ρ is made up by a family of linear maps $\rho_n : P(n) \to \mathcal{E}nd(V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$. The map ρ_n associates to abstract n-ary operations $\theta \in P(n)$ concrete n-ary operations on V, that is $\rho_n(\theta) \in \mathrm{Hom}(V^{\otimes n}, V)$. Therefore one can actually get an algebraic structure on V. A representation of the operad P on a vector space V endows it with corresponding algebraic structure, that called *P-algebra* or *algebra over P*.
Note that the linear maps $\rho_n : P(n) \to \mathcal{E}nd(V)(n)$ can also be viewed as

$$\rho_n \in \mathrm{Hom}(P(n), (\mathrm{Hom}(V^{\otimes n}, V)) = \mathrm{Hom}(P(n) \otimes V^{\otimes n}, V)$$
$$\rho_n : \theta \otimes v_1 \otimes ... \otimes v_n \to \theta(v_1, ..., v_n).$$

From the fact that the maps ρ_n respect the action of the permutation group follows that the space may be restricted to the smaller one

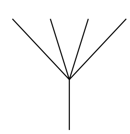
$$\rho_n \in \mathrm{Hom}(P(n) \otimes_{S_n} V^{\otimes n}, V) \subset \mathrm{Hom}(P(n) \otimes V^{\otimes n}, V). \tag{2.17}$$

Definition 2.3.10. The morphism between two P-algebras over V and W is a linear map $f : V \to W$, such that for any $n > 0$ and for any operation $\theta \in P(n)$ the following identity holds:

$$\theta(fv_1, ..., fv_n) = f\theta(v_1, ..., v_n).$$

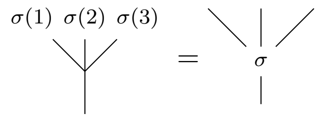
Now we are ready to consider other examples of operads

Example 2.3.11. The operad $\mathcal{C}om$ is the operad associated with commutative associative algebras. The vector spaces $\mathcal{C}om(n)$ are one dimensional for each $n > 0$ and $\mathcal{C}om(0)$=0. Pictorially:

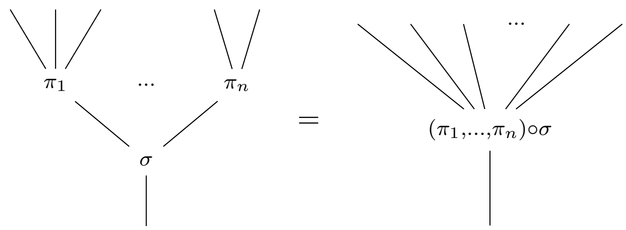
 $\in \mathcal{C}om(4)$

The composition and the action of the symmetric group are trivial. Any algebra over operad $\mathcal{C}om$ is a commutative associative algebra.

Example 2.3.12. The operad $\mathcal{A}ss$ associated with associative algebras, where $\mathcal{A}ss(n) = K[S_n]$ and $\mathcal{A}ss(0) = 0$. For example for $n = 3$ there are six basis elements in $\mathcal{A}ss(3)$:



The composition is given as indicated here:



Any algebra over operad $\mathcal{A}ss$ is an associative algebra.

2.3.3 S-modules. Schur functor

To the arbitrary S-module P on can correspond the Schur functor $\widetilde{P} : \mathrm{Vect} \to \mathrm{Vect}$:

$$\widetilde{P}(V) = \bigoplus_{n \in \mathbb{N}} P(n) \otimes_{S_n} V^{\otimes n},$$
$$\widetilde{P}(\ell) = \bigoplus_{n \in \mathbb{N}} \mathrm{id} \otimes_{S_n} \ell^{\otimes n} : \widetilde{P}(V) \to \widetilde{P}(W).$$

for any vector space V and any linear map $\ell : V \to W$.

The *composite* of two S-modules P and Q is the S-module $P \circ Q$ defined by

$$(P \circ Q)(n) := \bigoplus_{k \geqslant 0} P(k) \otimes_{S_k} \left(\bigoplus_{i_1 + ... + i_k = n} \big(Q(i_1) \otimes ... \otimes Q(i_k)\big) \otimes K[S_n / S_{i_1} \times ... \times S_{i_k}] \right)$$
$$= \bigoplus_{k \geqslant 0} P(k) \otimes_{S_k} \left(\bigoplus_{i_1 + ... + i_k = n} \big(Q(i_1) \otimes ... \otimes Q(i_k)\big) \otimes Sh(i_1, ..., i_k) \right).$$

Where the set $Sh(i_1, ..., i_k) \simeq S_n / S_{i_1} \times ... \times S_{i_k}$ of unshuffled permutations is defined in Definition 2.1.26. We denote the elements of $(P \circ Q)(n)$ by $(\mu; \nu_1, ..., \nu_k; \sigma)$, where $\mu \in P(k)$, $\nu_j \in Q(i_j)$ and $\sigma \in Sh(i_1, ..., i_k)$.
It is easy to check that the S-module $I = (0, K, 0, 0, ...)$ is the identity for the defined above composition. Note that the composition is additive only on the left factor, that is

$$(P_1 \oplus P_2) \circ Q = (P_1 \circ Q) \oplus (P_2 \circ Q),$$
$$P \circ (Q_1 \oplus Q_2) \neq (P \circ Q_1) \oplus (P \circ Q_2).$$

Proposition 2.3.13. *The Schur functor is compatible with the composition of S-modules, that is*

$$\widetilde{P} \circ \widetilde{Q} = \widetilde{P \circ Q}.$$

Proof. For an arbitrary vector space V the following identities holds:

$$\begin{aligned}
\widetilde{P \circ Q}(V) &= \bigoplus_{k,n \geqslant 0} P(k) \otimes_{S_k} \left(\bigoplus_{i_1+\ldots i_k = n} \Big(Q(i_1) \otimes \ldots \otimes Q(i_k) \Big) \otimes K[S_n/S_{i_1} \times \ldots \times S_{i_k}] \right) \otimes_{S_n} V^{\otimes n} \\
&= \bigoplus_{k,n \geqslant 0} P(k) \otimes_{S_k} \left(\bigoplus_{i_1+\ldots i_k = n} \Big(Q(i_1) \otimes \ldots \otimes Q(i_k) \Big) \otimes_{S_{i_1} \times \ldots \times S_{i_k}} K[S_n] \right) \otimes_{S_n} V^{\otimes n} \\
&= \bigoplus_{k,n \geqslant 0} P(k) \otimes_{S_k} \left(\bigoplus_{i_1+\ldots i_k = n} \Big(Q(i_1) \otimes \ldots \otimes Q(i_k) \Big) \otimes_{S_{i_1} \times \ldots \times S_{i_k}} V^{\otimes (i_1 + \ldots + i_k)} \right) \\
&= \bigoplus_{k,n \geqslant 0} P(k) \otimes_{S_k} \left(\bigoplus_{i_1+\ldots i_k = n} \Big(Q(i_1) \otimes_{S_{i_1}} V^{\otimes i_1} \Big) \otimes \ldots \otimes \Big(Q(i_k) \otimes_{S_{i_k}} V^{\otimes i_k} \Big) \right) \\
&= \bigoplus_{k \geqslant 0} P(k) \otimes_{S_k} \widetilde{Q}(V)^{\otimes k} = \widetilde{P}(\widetilde{Q}(V)) = \widetilde{P} \circ \widetilde{Q}(V).
\end{aligned}$$

□

The composition of the Schur functors is associative, that is

$$(\widetilde{P} \circ \widetilde{Q}) \circ \widetilde{R} \simeq \widetilde{P} \circ (\widetilde{Q} \circ \widetilde{R})$$

it implies that the composition of S-modules is associative too.

Definition 2.3.14. A S-module morphism $f : P \to P'$ consists of a sequence of linear maps $f_n : P(n) \xrightarrow{f} P'(n), n \geqslant 0$ which respect the action of the symmetric group:

$$f_n \sigma = \sigma f_n,$$

where σ is an arbitrary permutation in S_n.

Any morphism of S-modules $f : P \to P'$ gives rise to a natural transformation of Schur functors $\widetilde{f} : \widetilde{P} \to \widetilde{P'}$:

$$\widetilde{f}(V) : P(n) \otimes_{S_n} V^{\otimes n} \xrightarrow{f_n \otimes \mathrm{id}^{\otimes n}} P'(n) \otimes_{S_n} V^{\otimes n}.$$

For any S-module maps $f : P \to P'$ and $g : Q \to Q'$ the composition $f \circ g : P \circ Q \to P' \circ Q'$ is defined as follows:

$$(f \circ g)(P \circ Q) = fP \circ gQ$$

The category of S-modules with a defined above composition $\circ$ and with a identity object $I = (0, K, 0, 0, \ldots)$ form a monoidal category.

2.3.4 Functorial definition of operads

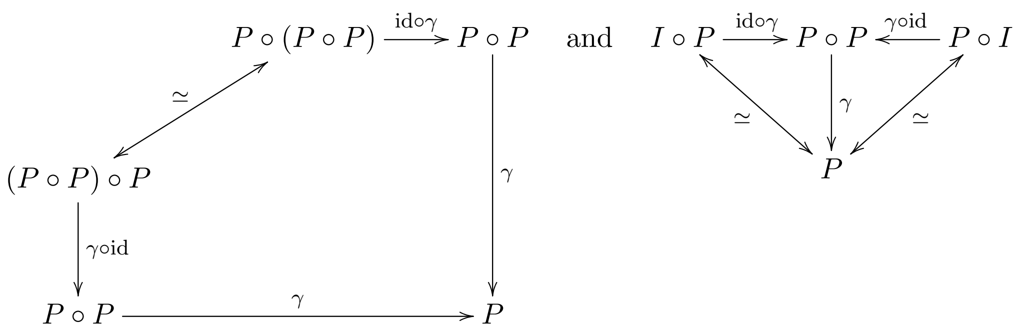
Definition 2.3.15. An *operad* $P = (P, \gamma, \eta)$ is an S-module P, endowed with morphsims of S-modules

$$\gamma : P \circ P \to P,$$

called *composition map*, and

$$\eta : I \to P,$$

called the *unit map*, such that the following diagrams commute:



(2.18)

Remark 2.3.16. The functorial and the classical definition of operad are equivalent. Indeed, for any element $(\theta; \theta_1, ..., \theta_n; \sigma) \in P \circ P$ the composition map $\gamma(\theta; \theta_1, ..., \theta_n; \sigma) \in P$ corresponds to the classical composition, then acting by σ from the right: $\theta \circ (\theta_1, ..., \theta_n)\sigma$. The identity element in the classical definition is $\eta(I) \in P(1)$. The equivariance axiom in the classical definition is equivalent to the fact that the corresponding elements in the functorial definition belong to the same equivalence class in $P \circ P$. The associativity and identity axioms correspond to the commutative diagrams above.

Remark 2.3.17. The functorial definition of operad is similar to the definition of the associative algebra. Indeed the associative algebra is a *monoid* in the monoidal category $(\text{Vect}, \otimes, K)$, that is an object $A \in \text{Vect}$ together with two morphisms: composition and identity morphism, that satisfy the associativity and unity requirements. The operad is a monoid in the monoidal category of S-modules $(\text{S-mod}, \circ, I)$. The obstacle to make the full analogy between associative algebras and operads is the fact that the composition $\circ$ of S-modules is not additive on the right factor. Nevertheless it is possible transfer some results from the theory on associative algebras to operads.

Now we give the functorial definition of operad morphism, which is equivalent to the classical one.

Definition 2.3.18. A *morphism of operads* from P to Q is a morphism of S-modules $\alpha : P \to Q$ which is compatible with the composition maps, that is

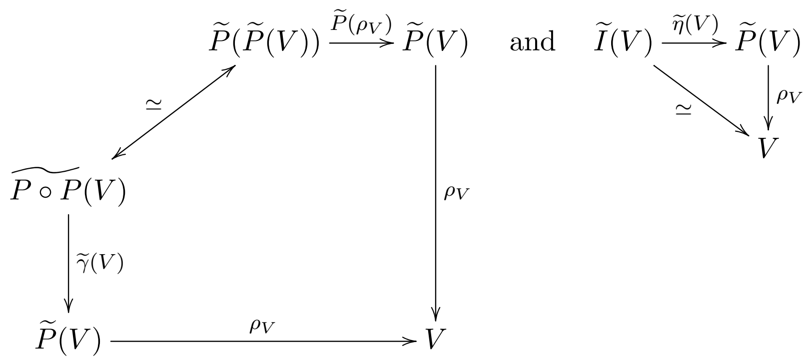
$$\gamma(\alpha P \circ \alpha P) = \alpha\gamma(P \circ P),$$

$$\alpha\eta_P = \eta_Q.$$

2.3.5 Functorial definition of P-algebras

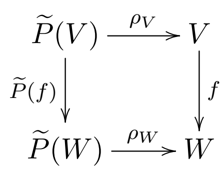
The notion of P-algebra can be given in the functorial language. Recall that in the classical framework a P-algebra is given by a representation ρ_V of operad P in $\mathcal{E}nd(V)$, which is given by the map $\bigoplus_{n\geqslant 0} P(n)\otimes_{S_n} V^{\otimes n} = \widetilde{P}(V) \xrightarrow{\rho_V} V$ (see Remark 2.3.9).

Definition 2.3.19. An *algebra over the operad P (or P-algebra)* is a vector space V equipped with a linear map $\rho_V : \widetilde{P}(V) \to V$ such that the following diagrams commute:



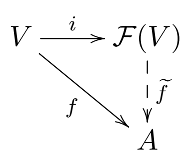
The notion of P-algebra morphism easily can be translated in the functorial language.

Definition 2.3.20. A morphism of P-algebras (V, ρ_V) and (W, ρ_W) is a linear map $f : V \to W$ such that the following diagram commutes:

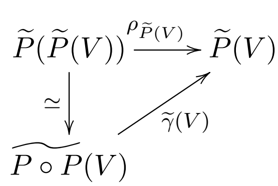


2.3.6 Free P-algebra

Definition 2.3.21. In the category of P-algebras, a P-algebra $\mathcal{F}(V)$, equipped with a linear map $i : V \to \mathcal{F}(V)$ is said to be *free* over the vector space V if for any P-algebra A and any linear map $f : V \to A$ there is a unique P-algebra morphism $\widetilde{f} : \mathcal{F}(V) \to A$ such that the following diagram is commutative:



Note that the free P-algebra is unique up to isomorphism. One can show that it is the vector space $\widetilde{P}(V)$ equipped with $\rho_{\widetilde{P}(V)} : \widetilde{P}(\widetilde{P}(V)) \to \widetilde{P}(V)$ defined via the diagram:



and the map $i : V \to \widetilde{P}(V)$, which is defined via the natural embedding

$$\begin{array}{ccc} V & \xrightarrow{\;i\;} & \widetilde{P}(V) \\ \simeq \downarrow & \nearrow_{\widetilde{\eta}(V)} & \\ \widetilde{I}(V) & & \end{array}$$

Example 2.3.22. For the operad $\mathcal{A}ss$, that controls associative algebras, the free $\mathcal{A}ss$-algebra over the vector space V is

$$\widetilde{\mathcal{A}ss}(V) = \bigoplus_{n>0} \mathcal{A}ss(n) \otimes_{S_n} V^{\otimes n} = \bigoplus_{n>0} K[S_n] \otimes_{S_n} V^{\otimes n} = \bigoplus_{n>0} V^{\otimes n} = \overline{T}(V),$$

where the composition is given by concatenation.

Example 2.3.23. For the operad $\mathcal{C}om$, that controls commutative associative algebras, the free $\mathcal{C}om$-algebra over the vector space V is

$$\widetilde{\mathcal{C}om}(V) = \bigoplus_{n>0} K \otimes_{S_n} V^{\otimes n} = \overline{S}(V),$$

where the composition is given by concatenation.

2.3.7 Free operad

As for any free object, the free operad over an S-module M is defined by means of a universal property. Namely, as being the operad $F(M)$ together with the S-module morphism $i : M \to F(M)$, such that for any operad P and any S-module morphism $\varphi : M \to P$, there exists a unique morphism of operads $\overline{\varphi} : F(M) \to P$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\;i\;} & F(M) \\ & {}_{\varphi}\searrow & \downarrow \widetilde{\varphi} \\ & & P \end{array}$$

In order to construct the free operad over M, we will define the sequence of $S-modules$ $\mathcal{T}_n M$ by

$$\begin{aligned} &\mathcal{T}_0 M = I, \\ &\mathcal{T}_1 M = I \oplus M, \\ &\mathcal{T}_2 M = I \oplus (M \circ (I \oplus M)) = I \oplus (M \circ \mathcal{T}_1 M), \\ &\quad \ldots \\ &\mathcal{T}_n M = I \oplus (M \circ \mathcal{T}_{n-1} M). \end{aligned}$$

Moreover, we recursively define a sequence of inclusions

$$\begin{aligned} &i_0 : \mathcal{T}_0 \to \mathcal{T}_n, \quad I \hookrightarrow I \oplus M; \\ &i_n : \mathcal{T}_{n-1} \to \mathcal{T}_n, \quad i_n = \mathrm{id}_I \oplus (\mathrm{id}_M \circ i_{n-1}). \end{aligned}$$

By definition, the *tree module* $\mathcal{T}M$ over the S-module M is a union

$$\mathcal{T}M := \bigcup_n \mathcal{T}_n M.$$

Remark 2.3.24. To get the intuition how the S-module $\mathcal{T}M$ is builded, it is useful to look at its elements as on formal operations:

$$\theta(....) = \Big\{a\Big(b(...)...c(....d(...)..)...\Big);\sigma\Big\}.$$

where the operations $a, b, c, d, ... \in M$ and the permutation σ permutes the elements in entries. If the 'depth' of the formal operation is equal to n then it belongs to $\mathcal{T}_n M$.

The obvious composition of these formal operations and the obvious unit operation define a *free operad* $\mathcal{T}M$.

The construction of a free operad gives a conceptual approach for any type of algebras to define the operad that controls these algebras. Let us be more precise: any type of algebraic structure on a vector space V is defined by the n-ry operations $A^{\otimes n} \to A$, called *generating operations*, that satisfy some specific multilinear relations . For example, associative algebras are given by binary multilinear map $\mu : A^{\otimes 2} \to A$ that satisfies $\mu \circ (\mu, \mathrm{id}) - \mu \circ (\mu, \mathrm{id}) = 0$. Let M be the S-module, whose arity n spaces are generated by the n-ary generating operations. Since the relators are composites of these generating relations (and identity), they span a sub-S-module of the free operad $\mathcal{T}M$. Let (R) denote the operadic ideal of $\mathcal{T}M$ generated by R. The precise definition of operadic ideals is given as follows:

Definition 2.3.25. An *operadic ideal* I of an operad P is a sub-S-module of P, such that for any family of operations $(\mu, \nu_1, ..., \nu_k)$ of P, we have that if one of these operations is in I, then the composite $\mu \circ (\nu_1, ..., \nu_k)$ is also in I.

This way, we have naturally constructed the operad $\mathcal{T}M/(R)$, which controls the type of algebra defined above (operations in M and relations R), that is any representation of $\mathcal{T}M/(R)$ in $\mathcal{E}nd(V)$ (any $\mathcal{T}M/(R)$-algebra on V) defines these types of algebras.

2.4 Operadic homological algebra

2.4.1 Infinitesimal composite

Recall that the composition of S-modules is not linear on the right factor. Here we introduce the infinitesimal composite, which is linear on two factors.

Consider the S-module $P \circ (I \oplus Q)$, its elements are of the form $(\mu; \nu_1, \mathrm{id}, \mathrm{id}, \nu_2, \mathrm{id}, ...; \sigma)$.

Definition 2.4.1. The *infinitesimal composite* $P \circ_{(1)} Q$ is the S-module, which is spanned by the elements of the 'linear part' of $P \circ (I \oplus Q)$, that is made up by the terms containing Q exactly once. The elements are of the form $(\mu; \mathrm{id}, ..., \mathrm{id}, \nu, \mathrm{id}, ..., \mathrm{id}; \sigma)$.

Note that the infinitesimal composite $\circ_{(1)}$ is not associative.

Definition 2.4.2. The corresponding composite $f \circ_{(1)} g$ of two $S-module$ morphisms $f : P_1 \to P_2$ and $g : Q_1 \to Q_2$ is defined by

$$f \circ_{(1)} g : P_1 \circ_{(1)} Q_1 \to P_2 \circ_{(1)} Q_2$$
$$(\mu; \mathrm{id}, ..., \mathrm{id}, \nu, \mathrm{id}, ..., \mathrm{id}; \sigma) \to (f(\mu); \mathrm{id}, ..., \mathrm{id}, g(\nu), \mathrm{id}, ..., \mathrm{id}; \sigma).$$

Instead of 'linearizing' the space $P \circ Q$, we can as well 'linearize' the morphism $f \circ g$:

Definition 2.4.3. The *infinitesimal composite* $f \circ' g$ of two $S-module$ morphisms $f : P_1 \to P_2$ and $g : Q_1 \to Q_2$ is defined by

$$\begin{aligned} f \circ' g : P_1 \circ Q_1 &\to P_2 \circ Q_2 \\ (\mu; \nu_1, ..., \nu_n; \sigma) &\to \sum_{i=1}^{n} (f(\mu); \nu_1, ..., g(\nu_i), ..., \nu_n; \sigma). \end{aligned}$$

2.4.2 Differential graded S-modules

Definition 2.4.4. A *graded S-module* P is a S-module such that its components $P(n)$ are graded vector spaces and the S_n action on them is degree preserving.

The subspace of $P(n)$ of the elements of degree k and we denote by $P_k(n)$.

Definition 2.4.5. A morphism $f : P \to Q$ of degree r between graded S-modules P and Q is a sequence of r-degree S_n-equivariant maps $f_n : P_k(n) \to Q_{k+r}(n), \forall k$.

Remark 2.4.6. The composite product $\circ$ can be extended to graded S-modules by

$$(P \circ Q)_p(n) := \bigoplus_{k \geqslant 0} P_r(k) \otimes_{S_k} \left(\bigoplus_{\substack{i_1+...+i_k=n \\ s_1+...+s_k=p-r}} \big(Q_{s_1}(i_1) \otimes ... \otimes Q_{s_k}(i_k)\big) \otimes K[S_n/S_{i_1} \times ... \times S_{i_k}] \right).$$

Definition 2.4.7. A *DG (differential graded) S-module* (P, d) is a graded S-module P endowed with a differential d, that is an S-module morphism $d : P \to P$ of degree -1, such that $d^2 = 0$.

Definition 2.4.8. A *morphism $f : (P, d_P) \to (Q, d_Q)$ of DG S-modules* is a graded S-module morphism $f : P \to Q$ of degree 0, such that it commutes with the differentials, that is

$$d_Q f = f d_P.$$

Definition 2.4.9. The composite of two DG S-modules (P, d_P) and (Q, d_Q) is the DG S-module $(P \circ Q, d_{P \circ Q})$, where the differential

$$d_{P \circ Q} := d_P \circ \mathrm{id}_Q + \mathrm{id}_P \circ' d_Q.$$

2.5 Differential graded operads and cooperads

Definition 2.5.1. A *DG (differential graded) operad* is a DG S-module (P, d_P) together with DG S-module morphisms: composition $\gamma : P \circ P \to P$ and unit map $\eta : I \to P$, which satisfy the associativity and unital relations (2.18).

Remark 2.5.2. The condition that the composition map γ is a DG S-module morphism means that it is a morphism of degree 0, such that

$$d_P \gamma = \gamma d_{P \circ P} = \gamma(\mathrm{id}_P \circ d_P + d_P \circ' \mathrm{id}_P).$$

Definition 2.5.3. A *DG (differential graded) cooperad* is a DG S-module (C, d_C) together with DG S-module morphisms: decomposition $\Delta : C \to C \circ C$ and counit map $\varepsilon : C \to I$, which satisfy the coassociativity and counital relations:

$$\begin{array}{ccc} & C \circ (C \circ C) & \xleftarrow{\mathrm{id}\circ\Delta} & C \circ C \\ \swarrow\!\nearrow \simeq & & & \\ (C \circ C) \circ C & & & \uparrow \Delta \\ \uparrow \Delta\circ\mathrm{id} & & & \\ C \circ C & \xleftarrow{\quad\Delta\quad} & & C \end{array} \quad \text{and} \quad \begin{array}{ccccc} I \circ C & \xleftarrow{\mathrm{id}\circ\Delta} & C \circ C & \xrightarrow{\Delta\circ\mathrm{id}} & C \circ I \\ & \simeq\nwarrow\!\searrow & \uparrow \Delta & \swarrow\!\nearrow\simeq & \\ & & P & & \end{array}$$

Remark 2.5.4. (DG) operads and cooperads are dual to each other, namely for any (DG) cooperad C there exists a (DG) operad on the S-module C^*, where the composition, unit, and differential are the transpositions of the corresponding 'co-morphisms'. In the other way round, for any operad P, where each $P(n)$ is finite dimensional one can construct the cooperad on P^* via the transposition of all structure maps. Note that to construct the operads from cooperads one not need to make the finite dimensional assumption. The explanation is similar to the explanation of the analogous statement in the algebraic world, see Remark 2.1.21.

2.5.1 Operadic twisting morphisms

To extend the theory of twisting morphisms to operads, we need the linearization of the composition map $\gamma : P \circ P \to P$ of an operad, and of the decomposition map $\Delta : C \to C \circ C$ of a cooperad.

Definition 2.5.5. The *infinitesimal composition map* of a (DG) operad is given by

$$\gamma_{(1)} : \; P \circ_{(1)} P \rightarrowtail P \circ P \xrightarrow{\gamma} P.$$

Definition 2.5.6. The *infinitesimal decomposition map* of a (DG) cooperad $\Delta_{(1)} : C \to C \circ_{(1)} C$ is given by

$$\Delta_{(1)} : \; C \xrightarrow{\Delta} C \circ C \twoheadrightarrow C \circ_{(1)} C.$$

From now on, we will require the DG operad (P, d_P, γ, η) to be *augmented*, that is there exists a DG S-module morphism $\varepsilon : P \to I$, that respects composition γ and unit η. Similarly the DG cooperad $(C, d_C, \Delta, \varepsilon)$ to be *coaugmented*, that is there exists a DG S-module morphism $\eta : I \to C$, that respects decomposition Δ and counit ε.

Definition 2.5.7. The *differential convolution algebra* on the space of graded S-module morphisms $\mathrm{Hom}_S(C, P) := \bigoplus_{n \geqslant 0} \mathrm{Hom}_{S_n}(C(n), P(n))$ is given by

$$f \star g : \; C \xrightarrow{\Delta_{(1)}} C \circ_{(1)} C \xrightarrow{f \circ_{(1)} g} P \circ_{(1)} P \xrightarrow{\gamma_{(1)}} P.$$

the unit of the product $\star$ is given by $\gamma \circ \varepsilon \in \mathrm{Hom}_S(C, P)$. And the differential

$$\partial f = d_P \circ f - (-1)^{\overline{f}} f \circ d_C.$$

Definition 2.5.8. An operadic twisting morphism $\alpha \in \mathrm{Tw}(C,P)$ is a -1-degree morphism in $\mathrm{Hom}_S(C,P)$ which is the solution of the Maurer-Cartan equation

$$\partial\alpha + \alpha \star \alpha = 0,$$

and is null when composed with the augmentation of P and also when with the coaugmentation of C.

Consider the DG S-module $(C \circ P, d_{C\circ P})$, where the differential $d_{C\circ P} = d_C \circ \mathrm{id}_P + \mathrm{id}_C \circ' d_P$. Moreover, consider graded S-module morphism $\alpha \in \mathrm{Hom}_S(C,P)$ and define $\overline{d}_\alpha : C\circ P \to C \circ P$ by the following diagram:

$$C \circ P \xrightarrow{\Delta_{(1)}\circ \mathrm{id}_P} (C \circ_{(1)} C) \circ P \xrightarrow{(\mathrm{id}_C \circ_{(1)} \alpha)\circ \mathrm{id}_P} (C \circ_{(1)} P) \circ P \rightarrowtail C \circ P \circ P \xrightarrow{\mathrm{id}_C \circ \gamma} C \circ P.$$

If $d_\alpha = d_{C\circ P} + \overline{d}_\alpha$ defines a differential, that is $d_\alpha^2 = 0$, which is the case if and only if α satisfies the Maurer-Cartan equation, then $C \circ_\alpha P := (C \circ P, d_\alpha)$ is a DG S-module called *twisted composite complex*. The *comparison lemma* remains valid for twisted composite complexes.

2.5.2 Operadic bar and cobar constructions

These constructions are similar to the corresponding ones in the algebraic context. Let us detail the cobar construction. Consider an augmented DG cooperad $(C, \Delta, \varepsilon, d_C)$, that is, in particular, we have $C = I \oplus \overline{C}$. Consider the free operad $\mathcal{T}(s^{-1}C)$, with the differential on it given by the sum $\delta_1 + \delta_2$, where δ_1 extends the differential d_C and δ_2 extends the infinitesimal decomposition $\Delta_{(1)}$. More precisely,

$$s^{-1}\overline{C} \xrightarrow{s} \overline{C} \xrightarrow{d_C} \overline{C} \xrightarrow{s^{-1}} s^{-1}\overline{C} \rightarrowtail \mathcal{T}(s^{-1}\overline{C})$$

and

$$s^{-1}\overline{C} \xrightarrow{s} \overline{C} \xrightarrow{\Delta_{(1)}} \overline{C} \circ_{(1)} \overline{C} \xrightarrow{s^{-1}\circ s^{-1}} s^{-1}\overline{C} \circ_{(1)} s^{-1}\overline{C} \rightarrowtail \mathcal{T}(s^{-1}\overline{C})$$

uniquely extend, since $\mathcal{T}(s^{-1}C)$ is free, to derivations δ_1 and δ_2 of $\mathcal{T}(s^{-1}C)$ and form the cobar complex $\Omega C = (\mathcal{T}(s^{-1}C), \delta_1 + \delta_2)$. Similarly with the algebraic context the following isomorphisms holds:

$$\mathrm{Hom}_{DG\ Op}(\Omega C, P) \simeq \mathrm{T}w(C,P) \simeq \mathrm{Hom}_{DG\ CoOp}(C, BP).$$

Under some weight-graded assumptions (similar to the one from associative algebras and coalgebras), we have:

$$\mathrm{QIso}_{DG\ Op}(\Omega C, P) \simeq \mathrm{Kos}(C,P) \simeq \mathrm{QIso}_{DG\ CoOp}(C, BP),$$

where $\alpha \in \mathrm{Kos}(C,P) \Leftrightarrow C \circ_\alpha P$ is acyclic. And taking $C = BP$, we find that $\Omega BP \xrightarrow{\sim} P$.

2.5.3 Koszul duality for operads

We will adapt the results of Koszul duality for algebras to operads. This will lead, for a quadratic Koszul operad P, to a model $P_\infty := \Omega P^{\text{¡}}$, which then allows to define P_∞-algebras (or homotopy P-algebras) as representations of this operad.

Definition 2.5.9. Operadic quadratic data (E, R) consists of a graded S-module E and a graded sub-S-module $R \in \mathcal{T}(E)^{(2)}$.

Here $\mathcal{T}(E)^{(2)}$ refers to the weight 2 part of the free operad $\mathcal{T}(E)$, that is to the graded sub-S-module of $\mathcal{T}(E)$, which is spanned by composites of two elements of E.

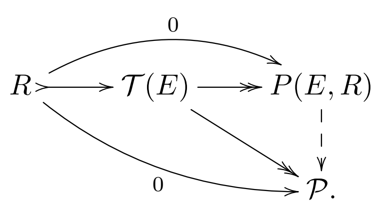
We will use the same terminology as in the algebraic setting and refer to elements of E as generating operations and to elements of R as relations.

Definition 2.5.10. The *quadratic operad* $P(E, R)$ associated to the operadic quadratic data (E, R) is the quotient operad $\mathcal{T}(E)/(R)$, where (R) denotes the operadic ideal generated by $R \subset \mathcal{T}(E)^{(2)}$.

The quadratic operad $P(E, R)$ is the quotient operad of $\mathcal{T}(E)$ that is universal among all quotient operads $\mathcal{P}$ of $\mathcal{T}(E)$, such that the composite

$$R \rightarrowtail \mathcal{T}(E) \twoheadrightarrow \mathcal{P}$$

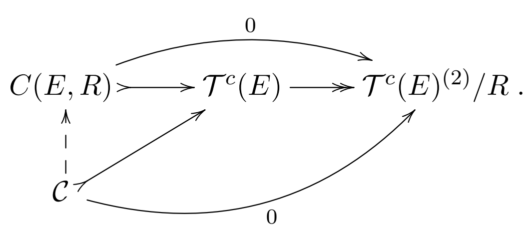
vanishes. More precisely, there exists a unique morphism of operads $P(E, R) \to \mathcal{P}$, such that the following diagram commutes



Definition 2.5.11. The *quadratic cooperad* $C(E, R)$ associated to the operadic quadratic data (E, R) is the subcooperad of the cofree cooperad $\mathcal{T}^c(E)$, that is universal among all subcooperads $\mathcal{C}$ of $\mathcal{T}^c(E)$, such that the composite

$$\mathcal{C} \rightarrowtail \mathcal{T}^c(E) \twoheadrightarrow \mathcal{T}^c(E)^{(2)}/R$$

vanishes. More precisely, there exists a unique morphism of cooperads $\mathcal{C} \to C(E, R)$, such that the following diagram commutes



Note that when we are working over graded S-modules, the above defined quadratic operad (respectively cooperad) is not only endowed with an arity grading and a weight grading (coming from the free, respectively, cofree operad), but also with a degree.

2.5.4 Koszul dual cooperad and operad of a quadratic operad

Definition 2.5.12. The *Koszul dual cooperad* of a quadratic operad $P = P(E, R)$ is

$$P^{¡} = C(sE, s^2R),$$

that is the quadratic cooperad associated to the shifted operadic quadratic data.

Here sE denotes the shifted S-module, obtained from E by shifting the degree in each arity.

In order to define the Koszul dual operad, we need some preliminary remarks.
First, the *Hadamard product* $P \underset{\mathrm{H}}{\otimes} Q$ of two S-modules is given by $(P \underset{\mathrm{H}}{\otimes} Q)(n) = P(n) \otimes Q(n)$. The action of the symmetric group is given by the diagonal action, i.e. $(\mu \otimes \nu) \cdot \sigma = (\mu \cdot \sigma) \otimes (\nu \cdot \sigma)$, for any $\mu \in P(n)$, $\nu \in Q(n)$, $\sigma \in S_n$. Moreover, the Hadamard product of operads has a natural operad structure.
Second, the suspension of an operad, obtained by suspending the underlying S-module, is, in general, not an operad. Therefore, we will define an 'operadic suspension'. Let $\mathcal{S} := \mathcal{E}nd(sK)$ be the endomorphism operad over the suspended ground field. This means that $\mathcal{S}(n) = \mathrm{Hom}((sK)^{\otimes n}, sK)$; note that this space contains morphisms of degree $-n+1$. The symmetric group action is given by the signature action. We also denote $\mathcal{S}^{-1} := \mathcal{E}nd(s^{-1}K)$ and $\mathcal{S}^c := \mathcal{E}nd^c(sK)$, where $\mathcal{E}nd^c(sK)$ is the endomorphism cooperad, which is as S-module the same as the endomorphism operad, but equipped with a decomposition map.
Finally, we define the *operadic suspension* of an operad P by $\mathcal{S} \underset{\mathrm{H}}{\otimes} P$. The *operadic desuspension* is given by $\mathcal{S}^{-1} \underset{\mathrm{H}}{\otimes} P$. For a cooperad C, the *cooperadic suspension* is given by $\mathcal{S}^c \underset{\mathrm{H}}{\otimes} C$. The operadic suspension has the property that a vector space V is equipped with a P-algebra structure, if and only if the suspended vector space sV is equipped with a $\mathcal{S} \underset{\mathrm{H}}{\otimes} P$-algebra structure.

Definition 2.5.13. The *Koszul dual operad* of a quadratic operad $P = P(E, R)$ is defined by

$$P^! = (\mathcal{S}^c \underset{\mathrm{H}}{\otimes} P^{¡})^*.$$

The dual means here that we take the linear dual in each arity.
Let us mention that the $P^!$ is quadratic in a certain case. More precisely,

Proposition 2.5.14. *Let $P = P(E, R)$ be a quadratic operad, generated by a reduced S-module E which is of finite dimension in each arity. Then the Koszul dual operad $P^!$ admits the quadratic presentation $P^! = P(s^{-1}\mathcal{S}^{-1} \underset{\mathrm{H}}{\otimes} E^*, R^{\perp})$.*

Moreover, we have that, under the assumptions of the previous proposition, $(P^!)^! = P$.

2.5.5 Koszul operads and infinity algebras

For given operadic quadratic data (E, R), we have that $P(E, R)^{(1)} = E$ and $C(E, R)^{(1)} = E$, and we can define the morphism κ by

$$\kappa : C(sE, s^2R) \twoheadrightarrow sE \overset{s^{-1}}{\to} E \rightarrowtail P(E, R).$$

This morphism is clearly of degree -1, and verifies (for the same reasons as in the algebraic case) $\kappa \star \kappa = 0$. Therefore, $\kappa \in \mathrm{Tw}$ is an operadic twisting morphism.
This defines a *Koszul complex* $P^{¡} \circ_\kappa P := (P^{¡} \circ P, d_\kappa)$. We thus have a sequence of chain complexes of S_n-modules $((P^{¡} \circ P)(n), d_\kappa)$, called Koszul complexes in arity n.
A quadratic operad P is called a *Koszul operad* if the corresponding Koszul complex $P^{¡} \circ_\kappa P$ is acyclic.

Let us mention that there exists many Koszul operads, in particular $\mathcal{A}ss$, $\mathcal{C}om$, $\mathcal{L}ie$, $\mathcal{L}eib$ and $\mathcal{P}ois$ are Koszul operads.

Just as we have for Koszul algebras A, a resolution $\Omega A^{¡} \xrightarrow{\sim} A$, we obtain, for Koszul operads P, a resolution $\Omega P^{¡} \xrightarrow{\sim} P$. The operad $\Omega P^{¡}$ is the P_∞-operad. Hence, to a P-algebra structure on a vector space V, given by $P \to \mathcal{E}nd(V)$, corresponds via

$$\begin{array}{ccc} P_\infty := \Omega P^{¡} & \xrightarrow{\sim} & P \\ & \searrow & \downarrow \\ & & \mathcal{E}nd(V) \end{array}$$

a P_∞-algebra (also called homotopy P-algebra) structure on V.

Theorem 2.5.15 (Ginzburg-Kapranov)**.** *[GK94]. Let P be a quadratic Koszul operad. A P_∞-structure on a graded vector space V, in the sense of a representation on V of the DG operad $P_\infty := \Omega P^{i}$, is equivalent (in the finite-dimensional setting) to a square-zero derivation of degree -1 on the free $P^!$-algebra over $s^{-1}V^*$*

$$P_\infty - \textit{algebra on } V \quad \Leftrightarrow \quad d \in \mathrm{Der}_{-1}(\mathrm{Free}_{P^!}(s^{-1}V^*)),\ d^2 = 0.$$

Similarly P_∞-structure on V (here, no finite-dimensional requirement is needed) is equivalent to the square-zero coderivation of degree -1 on the cofree $P^!$-coalgebra over sV

$$P_\infty - \textit{algebra on } V \quad \Leftrightarrow \quad D \in \mathrm{CoDer}_{-1}(\mathrm{CoFree}_{P^!}(sV)),\ D^2 = 0.$$

2.6 Lie infinity algebras

Usual differential graded Lie algebras can be considered as a very special case of L_∞ algebras. One may consider them as objects of the category of L_∞-algebras. Thus we radically increase morphisms. This has very important applications in mathematical physics, for example it was essentially used in the famous Kontsevich's proof of the existence of deformation quantization for any Poisson manifold.
Lie infinity algebras can be constructed from the general operadic approach. Let us be more precise: the quadratic operad $\mathcal{L}ie$, which controls graded Lie algebras is Koszul. Its Koszul dual operad $\mathcal{L}ie^! = \mathcal{C}om$ controls graded commutative associative algebras. Lie infinity operad $\mathcal{L}ie_\infty$, by definition, is the cobar construction $\Omega\mathcal{L}ie^{¡}$. The representation of the operad $\mathcal{L}ie_\infty$ on a vector space V defines L_∞ (Lie_∞) algebra on V. Due to the Ginzburg-Kapranov theorem (2.5.15) L_∞ algebra on V is equivalent to a square-zero coderivation D of degree -1 on the free $\mathcal{L}ie^! = \mathcal{C}om$ coalgebra over sV, that is the reduced symmetric coalgebra $\overline{S}^c(sV)$.

2.6.1 Definition of Lie infinity algebra

Here we give the classical definition of L_∞ algebra and then show that it coincides with the operadic one.

Definition 2.6.1. *L_∞ algebra on a graded vector space V* is given by the family of multilinear maps $l_i : V^{\otimes i} \to V$ of degrees $(i-2)$ s.t. the following conditions holds:
- *the graded antisymmetry*

$$l_i(v_1, ..., v_i) = \mathrm{sign}(\sigma) \cdot \varepsilon(\sigma) \cdot l_i(v_{\sigma(1)}, ..., v_{\sigma(i)}), \tag{2.19}$$

- *higher Jacobi identities.* For any $n > 0$:

$$\sum_{i+j-1=n} \sum_{\sigma \in Sh(i,j-1)} (-1)^{i(j-1)} \mathrm{sign}(\sigma) \cdot \varepsilon(\sigma) \cdot l_j(l_i(v_{\sigma(1)}, ..., v_{\sigma(i)}), v_{\sigma(i+1)}, ..., v_{\sigma(i+j-1)}). \tag{2.20}$$

Remark 2.6.2. One can reformulate the definition of L_∞ algebra and say that the structure maps $\{l_i\}$ are defined on the on the reduced exterior algebra $\overline{\Lambda}(V)$ and $l_i : \Lambda^i(V) \to V$. Then the graded antisymmetric conditions (2.19) hold automatically.

Proposition 2.6.3. *The operadic definition of L_∞ algebra is equivalent to the classical one. Namely L_∞ algebra (in the classical sense) over a finite dimensional graded vector space V is given by the differential d on the reduced symmetric algebra $\overline{S}(s^{-1}V^*)$ or dually by the codifferential $D = d^*$ on the reduced symmetric coalgebra $\overline{S}^c(sV)$. And the condition $d^2 = 0$ (dually $D^2 = 0$) encodes the bunch of higher Jacobi identities (2.20).*

Proof. Consider an arbitrary finite dimensional graded vector space W and the differential d on the reduced symmetric algebra $\overline{S}(W)$. The derivation d is given by its action on the generators, that is by the action on the vector space W. By applying the Leibniz rule one can get the action of the derivation on the whole space $\overline{S}(W)$ knowing only its action on W. For any element $w_1...w_p \in S^p(W)$:

$$\begin{aligned} d(w_1...w_p) &= \sum_{i=1}^{p} (-1)^{\overline{w_1}+...+\overline{w_{i-1}}} (w_1...dw_i...w_p) = \\ &= (d \odot \underbrace{\mathrm{id} \odot ... \odot \mathrm{id}}_{p-1})(w_1...w_p) = (d \odot \mathrm{id}^{\odot(p-1)})(w_1...w_p). \end{aligned} \tag{2.21}$$

If for an arbitrary -1-degree derivation d on $\overline{S}(W)$ the action of d^2 on the generators is zero then it is zero on the whole space $\overline{S}(W)$, and therefore the derivation becomes a differential. It follows from the following identity:

$$d^2(w \cdot v) = d^2 w \cdot v + w \cdot d^2 v = 0.$$

The action of the differential d on $\overline{S}(W)$ is given by its action on generators, that is the series of linear maps

$$d_p : W \to S^p(W),\ d|_W = d_1 + d_2 + ...$$

and the condition that $d^2 = 0$ reads as follows:

$$d(dw) = \sum_{p=1}^{\infty} d(d_p w) \overset{(2.21)}{=} \sum_{p=1}^{\infty} (d \odot \mathrm{id}^{\odot(p-1)}) d_p w =$$
$$= \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} (d_k \odot \mathrm{id}^{\odot(p-1)}) d_p w = \sum_{n=1}^{\infty} \sum_{p+k-1=n} (d_k \odot \mathrm{id}^{\odot(p-1)}) d_p w = 0.$$

In the dual language, the transposed map $D = d^*$ is a codifferential on the reduced symmetric coalgebra $\overline{S}^c(W^*)$. And the transposition of the last identity will encode that $D^2 = 0$:

$$\sum_{n=1}^{\infty} \sum_{p+k-1=n} D_p(D_k \odot \mathrm{id}^{\odot(p-1)}) = 0, \tag{2.22}$$

where the transposed maps $D_p = d_p^* : S^p(W^*) \to W^*$ called the corestrictions of the codifferential D.

In equation (2.22) the weight of the operator inside the sum $\sum_{n=1}^{\infty}$ is equal to $2 - p - k = 1 - n$, so it depends only on n, that is the last equation splits into the series of equations:

$$(D^2)_n = \sum_{p+k-1=n} D_p(D_k \odot \mathrm{id}^{\odot(p-1)}) = 0, \quad \text{where } n > 0. \tag{2.23}$$

The condition $D^2 = 0$ on the reduced symmetric coalgebra $\overline{S}^c(sV)$ reads as follows:

$$\sum_{p+k-1=n} D_p(D_k \odot \mathrm{id}^{\odot(p-1)})(sv_1 ... sv_n) = 0, \quad n > 0.$$

We choose the representative element $sv_1 \otimes ... \otimes sv_1 \in sv_1 ... sv_n$ then the last identity becomes

$$\sum_{p+k-1=n} D_p \circ (D_k \otimes \mathrm{id}^{\otimes(p-1)}) \circ \left(\sum_{\sigma \in Sh(k,p-1)} \sigma \right) (sv_1 \otimes ... \otimes sv_n) = 0, \quad n > 0.$$

Now we insert identities of the type $(-1)^{\frac{i(i-1)}{2}} s^{\otimes i} (s^{-1})^{\otimes i} = \mathrm{id}^{\otimes i}$ in two places:

$$\begin{aligned} &\sum_{p+k-1=n} \overbrace{s^{-1} D_p s^{\otimes p}}^{l_p} (-1)^{\frac{p(p-1)}{2}} \overbrace{\left(s^{-1}\right)^{\otimes p} (D_k \otimes \mathrm{id}^{\otimes(p-1)}) s^{\otimes n}}^{\pm l_k \otimes \mathrm{id}^{\otimes(p-1)}} \circ \\ &\circ (-1)^{\frac{n(n-1)}{2}} \underbrace{\left(s^{-1}\right)^{\otimes n} \left(\sum_{\sigma \in Sh(k,p-1)} \sigma \right) s^{\otimes n}}_{\sum_{\sigma \in Sh(k,p-1)} \pm \sigma} = 0, \quad n > 0. \end{aligned} \tag{2.24}$$

The precise signs in the formula above are the following:

1. $s^{-1} D_p s^{\otimes p} = l_p$,
2. $\left(s^{-1}\right)^{\otimes p} (D_k \otimes \mathrm{id}^{\otimes(p-1)}) s^{\otimes(p+k-1)} = (-1)^{\frac{p(p-1)}{2} + k(p-1)} l_k \otimes \mathrm{id}^{\otimes(p-1)}$,

3. $\left(s^{-1}\right)^{\otimes n}\left(\sum\limits_{\sigma\in Sh(k,p-1)}\sigma\right)s^{\otimes n}=\sum\limits_{\sigma\in Sh(k,p-1)}(-1)^{\frac{n(n-1)}{2}}\operatorname{sign}(\sigma)\cdot\sigma.$

We evaluate the operators (2.24) on the elements $v_1\otimes...\otimes v_n$ and get

$$\sum_{k+p-1=n}\sum_{\sigma\in Sh(k,p-1)}(-1)^{k(p-1)}\operatorname{sign}(\sigma)\cdot\varepsilon(\sigma)\cdot l_p(l_k(v_{\sigma(1)},...,v_{\sigma(k)}),v_{\sigma(k+1)},...,v_{\sigma(k+p-1)}).$$

The structure maps $l_i=D_i^{susp}$ have a degree $i-2$, and they act on the reduced exterior coalgebra $\overline{\Lambda}^c(V)$ (as a vector space is isomorphic to the reduced exterior algebra $\overline{\Lambda}(V)$). That guarantees that the graded antisymmetric conditions (2.19) hold true. □

Definition 2.6.4. If the degrees of the graded vector space V are concentrated from 0 to $n-1$: $V=V_0\oplus...\oplus V_{n-1}$ then the L_∞ structure on V is called *n-term L_∞-algebra*.

2.6.2 Lie infinity algebra morphism

Definition 2.6.5. The morphism between two L_∞ algebras over V and W is given by the family of multilinear maps $\varphi_i:\Lambda^i(V)\to W$ of degree $(i-1)$ which for any $n>0$ satisfy the following identities:

$$\begin{aligned}&\sum_{p=1}^{n}\sum_{k_1+...+k_p=n}\sum_{\sigma\in Sh(k_1,...,k_p)}(-1)^{\frac{p(p-1)}{2}+\sum\limits_{i=1}^{p-1}(p-i)k_i}\cdot(-1)^{\sum\limits_{i=2}^{p}(k_i-1)(\overline{v}_{\sigma(1)}+...+\overline{v}_{\sigma(k_1+...+k_{i-1})})}\times\\&\times\operatorname{sign}(\sigma)\cdot\varepsilon(\sigma)\cdot l_p(\varphi_{k_1}(v_{\sigma(1)},...,v_{\sigma(k_1)}),...,\varphi_{k_p}(v_{\sigma(k_1+...+k_{p-1}+1)},...,v_{\sigma(k_1+...+k_p)}))=\\&=\sum_{k+p-1=n}\sum_{\sigma\in Sh(k,p-1)}(-1)^{k(p-1)}\cdot\operatorname{sign}(\sigma)\cdot\varepsilon(\sigma)\cdot\varphi_p(l_k(v_{\sigma(1)},...,v_{\sigma(k)}),v_{\sigma(k+1)},...,v_{\sigma(k+p-1)}).\end{aligned}\tag{2.25}$$

Proposition 2.6.6. *L_∞-algebra morphism between L_∞-algebras over V and W is given by the differential algebra morphism $f:S(s^{-1}W^*)\to S(s^{-1}V^*)$, or dually by the codifferential coalgebra morphism $F=f^*:S^c(sV)\to S^c(sW)$. The morphism condition $fd-df=0$ (or dually $FD-DF=0$) encodes the bunch of higher identities (2.25).*

Proof. Consider an arbitrary graded vector spaces U and U'. An arbitrary symmetric algebra morphism $f:S(U)\to S(U')$ is given by its action on generators:

$$f(u_1...u_p)=f(u_1)\cdot...\cdot f(u_p)=\underbrace{(f\odot...\odot f)}_{p}(u_1...u_p).\tag{2.26}$$

If for the algebra morphism f the condition $df-fd=0$ holds on the generators then it also holds on the whole space $S(U)$, it follows from the fact that

$$(fd-df)(u\cdot w)=(fd-df)u\cdot w+(-1)^{\overline{u}}u\cdot(fd-df)w.$$

The action of the differential algebra morphism is given by the series of maps $f_p:U\to S^p(U')$, and the condition that the algebra morphism f is a differential algebra morphism reads as follows:

$$\begin{aligned}(fd-df)u&=f\sum_{p=1}^{\infty}d_pu-d\sum_{p=1}^{\infty}f_pu\overset{(2.21),(2.26)}{=}\sum_{p=1}^{\infty}f^{\odot p}d_pu-\sum_{p=1}^{\infty}(d\odot\mathrm{id}^{\odot(p-1)})f_pu=\\&=\sum_{n=1}^{\infty}\sum_{p=1}^{n}\sum_{k_1+...+k_p=n}(f_{k_1}\odot...\odot f_{k_p})d_pu-\sum_{n=1}^{\infty}\sum_{p+k-1=n}(d_k\odot\mathrm{id}^{\odot(p-1)})f_pu=0.\end{aligned}$$

In the dual language, the transposed map $F = f^*$ is a morphism of the reduced symmetric coalgebras $F : \overline{S}^c(U'^*) \to \overline{S}^c(U^*)$. And the transposition of the last identity encodes that $FD - DF = 0$:

$$\sum_{n=1}^{\infty}\left(\sum_{p=1}^{n}\sum_{k_1+...+k_p=n} D_p(F_{k_1}\odot ...\odot F_{k_p}) - \sum_{p+k-1=n}^{\infty} F_p(D_k\odot \mathrm{id}^{\odot(p-1)})\right) = 0, \tag{2.27}$$

where the transposed maps $F_k = f_k^* : S^k(U'^*) \to U^*$ are called the corestrictions of the morphism F.

In equation (2.27) the weight of the operator inside the sum $\sum\limits_{n=1}^{\infty}$ is equal to $1-n$, so it depends only on n, i.e. the last equation splits into the series of equations:

$$\sum_{p=1}^{n}\sum_{k_1+...+k_p=n} D_p(F_{k_1}\odot ...\odot F_{k_p}) - \sum_{p+k-1=n}^{\infty} F_p(D_k\odot \mathrm{id}^{\odot(p-1)}) = 0,\ \text{where } n>0. \tag{2.28}$$

We choose the representative element such that $[sv_1\otimes ...\otimes sv_1] = sv_1...sv_n \in S^n(sV)$. And the condition $FD - DF = 0 : \overline{S}^c(sV) \to \overline{S}^c(sW)$ reads as follows:

$$\sum_{p=1}^{n}\sum_{k_1+...+k_p=n} D_p(F_{k_1}\otimes ...\otimes F_{k_p})\circ\left(\sum_{\sigma\in Sh(k_1,...,k_p)}\sigma\right)(sv_1\otimes ...\otimes sv_n) =$$
$$\sum_{p+k-1=n}^{\infty} F_p\circ(D_k\otimes \mathrm{id}^{\otimes(p-1)})\circ\left(\sum_{\sigma\in Sh(k,p-1)}\sigma\right)(sv_1\otimes ...\otimes sv_n),\ n>0.$$

Now we insert in certain places the identities of the type $(-1)^{\frac{i(i-1)}{2}}s^{\otimes i}(s^{-1})^{\otimes i} = \mathrm{id}^{\otimes i}$:

$$\begin{aligned}&\sum_{p=1}^{n}\sum_{k_1+...+k_p=n}\overbrace{s^{-1}D_p s^{\otimes p}}^{l_p}(-1)^{\frac{p(p-1)}{2}}\overbrace{(s^{-1})^{\otimes p}(F_{k_1}\otimes ...\otimes F_{k_p})s^{\otimes n}}^{\pm\varphi_{k_1}\otimes ...\otimes\varphi_{k_n}}\circ\\ &(-1)^{\frac{n(n-1)}{2}}\circ\overbrace{(s^{-1})^{\otimes n}\left(\sum_{\sigma\in Sh(k_1,...,k_p)}\sigma\right)s^{\otimes n}}^{\sum_{\sigma\in Sh(k_1,...,k_p)}\pm\sigma} =\\ &=\sum_{p+k-1=n}^{\infty}\overbrace{s^{-1}F_p s^{\otimes p}}^{\varphi_p}(-1)^{\frac{p(p-1)}{2}}\overbrace{(s^{-1})^{\otimes p}(D_k\otimes \mathrm{id}^{\otimes(p-1)})s^{\otimes n}}^{\pm l_k\otimes \mathrm{id}^{\otimes(p-1)}}\circ\\ &\circ(-1)^{\frac{n(n-1)}{2}}\overbrace{(s^{-1})^{\otimes n}\left(\sum_{\sigma\in Sh(k,p-1)}\sigma\right)s^{\otimes n}}^{\sum_{\sigma\in Sh(k,p-1)}\pm\sigma},\ n>0.\end{aligned} \tag{2.29}$$

The precise signs in the formula above are the following:

1. $s^{-1}D_p s^{\otimes p} = l_p$,

2. $(s^{-1})^{\otimes p}(F_{k_1} \otimes ... \otimes F_{k_p})s^{\otimes(k_1+...+k_p)} = (-1)^{\sum\limits_{i=1}^{p-1}(p-i)k_i}(\varphi_{k_1} \otimes ... \otimes \varphi_{k_p})$,

3. $s^{-1}F_p s^{\otimes p} = \varphi_p$,

4. $\left(s^{-1}\right)^{\otimes p}(D_k \otimes \mathrm{id}^{\otimes(p-1)})s^{\otimes(p+k-1)} = (-1)^{\left[\frac{p(p-1)}{2}+k(p-1)\right]} \cdot l_k \otimes \mathrm{id}^{\otimes(p-1)}$,

5. $\left(s^{-1}\right)^{\otimes n}\sigma s^{\otimes n} = (-1)^{\frac{n(n-1)}{2}}\mathrm{sign}(\sigma) \cdot \sigma$.

We evaluate the operators (2.29) on the elements $v_1 \otimes ... \otimes v_n$ and get identities (2.25). $\square$

3 Higher categorified algebras versus bounded homotopy algebras

The following research paper was published in 'Theory and Applications of Categories', 25(10) (2011), 251-275 (joint work with Ashis Mandal and Norbert Poncin).

3.1 Introduction

Higher structures – infinity algebras and other objects up to homotopy, higher categories, "oidified" concepts, higher Lie theory, higher gauge theory... – are currently intensively investigated. In particular, higher generalizations of Lie algebras have been conceived under various names, e.g. Lie infinity algebras, Lie n-algebras, quasi-free differential graded commutative associative algebras (qfDGCAs for short), n-ary Lie algebras, see e.g. [Dz05], crossed modules [MP09] ...

More precisely, there are essentially two ways to increase the flexibility of an algebraic structure: homotopification and categorification.

Homotopy, sh or infinity algebras [Sta63] are homotopy invariant extensions of differential graded algebras. This property explains their origin in BRST of closed string field theory. One of the prominent applications of Lie infinity algebras [LS93] is their appearance in Deformation Quantization of Poisson manifolds. The deformation map can be extended from differential graded Lie algebras (DGLAs) to L_∞-algebras and more precisely to a functor from the category $\mathtt{L}_\infty$ to the category $\mathtt{Set}$. This functor transforms a weak equivalence into a bijection. When applied to the DGLAs of polyvector fields and polydifferential operators, the latter result, combined with the formality theorem, provides the 1-to-1 correspondence between Poisson tensors and star products.

On the other hand, categorification [CF94], [Cra95] is characterized by the replacement of sets (resp. maps, equations) by categories (resp. functors, natural isomorphisms). Rather than considering two maps as equal, one details a way of identifying them. Categorification is a sharpened viewpoint that leads to astonishing results in TFT, bosonic string theory... Categorified Lie algebras, i.e. Lie 2-algebras (alternatively, semistrict Lie 2-algebras) in the category theoretical sense, have been introduced by J. Baez and A. Crans [BC04]. Their generalization, weak Lie 2-algebras (alternatively, Lie 2-algebras), has been studied by D. Roytenberg [Roy07].

It has been shown in [BC04] that categorification and homotopification are tightly connected. To be exact, Lie 2-algebras and 2-term Lie infinity algebras form equivalent 2-categories. Due to this result, Lie n-algebras are often defined as sh Lie algebras concentrated in the first n degrees [Hen08]. However, this 'definition' is merely a terminological convention, see e.g. Definition 4 in [SS07]. On the other hand, Lie infinity algebra structures on an $\mathbb{N}$-graded vector space V are in 1-to-1 correspondence with square 0 degree -1 (with respect to the grading induced by V) coderivations of the free reduced graded commutative associative coalgebra $S^c(sV)$, where s denotes the suspension operator, see e.g. [SS07] or [GK94]. In finite dimension, the latter result admits a variant based on qfDGCAs instead of coalgebras. Higher morphisms of free DGCAs have been investigated under the name of derivation homotopies in [SS07]. Quite a number of examples can be found in [SS07b].

Besides the proof of the mentioned correspondence between Lie 2-algebras and 2-term Lie infinity algebras, the seminal work [BC04] provides a classification of all Lie infinity algebras, whose only

nontrivial terms are those of degree 0 and $n-1$, by means of a Lie algebra, a representation and an $(n+1)$-cohomology class; for a possible extension of this classification, see [Bae07].
In this paper, we give an explicit categorical definition of Lie 3-algebras and prove that these are in 1-to-1 correspondence with the 3-term Lie infinity algebras, whose bilinear and trilinear maps vanish in degree $(1,1)$ and in total degree 1, respectively. Note that a '3-term' Lie infinity algebra implemented by a 4-cocycle [BC04] is an example of a Lie 3-algebra in the sense of the present work.

The correspondence between categorified and bounded homotopy algebras is expected to involve classical functors and chain maps, like e.g. the normalization and Dold-Kan functors, the (lax and oplax monoidal) Eilenberg-Zilber and Alexander-Whitney chain maps, the nerve functor... We show that the challenge ultimately resides in an incompatibility of the cartesian product of linear n-categories with the monoidal structure of this category, thus answering a question of [Roy07].

The paper is organized as follows. Section 3.2 contains all relevant higher categorical definitions. In Section 3.3, we define Lie 3-algebras. Section 3.4 contains the proof of the mentioned 1-to-1 correspondence between categorified algebras and truncated sh algebras – the main result of this paper. A specific aspect of the monoidal structure of the category of linear n-categories is highlighted in Section 3.5. In Section 3.6, we show that this feature is an obstruction to the use of the Eilenberg-Zilber map in the proof of the correspondence "bracket functor – chain map".

3.2 Higher linear categories and bounded chain complexes of vector spaces

Let us emphasize that notation and terminology used in the present work originate in [BC04], [Roy07], as well as in [Lei04]. For instance, a linear n-category will be an (a strict) n-category [Lei04] in `Vect`. Categories in `Vect` have been considered in [BC04] and also called internal categories or 2-vector spaces. In [BC04], see Sections 2 and 3, the corresponding morphisms (resp. 2-morphisms) are termed linear functors (resp. linear natural transformations), and the resulting 2-category is denoted by `VectCat` and also by `2Vect`. Therefore, the $(n+1)$-category made up by linear n-categories (n-categories in `Vect` or $(n+1)$-vector spaces), linear n-functors... will be denoted by `Vect` n-`Cat` or $(n+1)$`Vect`.

The following result is known. We briefly explain it here as its proof and the involved concepts are important for an easy reading of this paper.

Proposition 3.2.1. *The categories* `Vect` n-`Cat` *of linear n-categories and linear n-functors and* $\mathtt{C}^{n+1}(\mathtt{Vect})$ *of $(n+1)$-term chain complexes of vector spaces and linear chain maps are equivalent.*

We first recall some definitions.

Definition 3.2.2. An **n-globular vector space** L, $n \in \mathbb{N}$, is a sequence

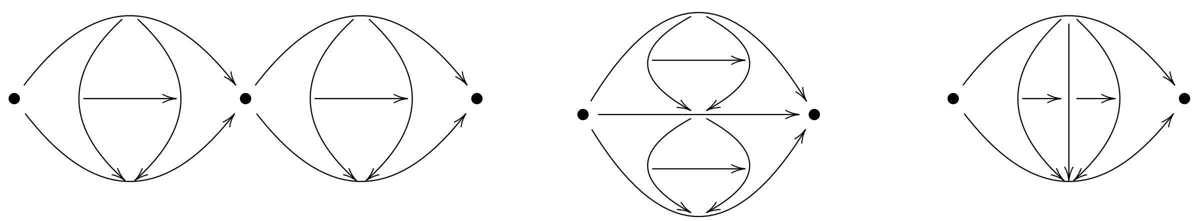
$$L_n \overset{s,t}{\rightrightarrows} L_{n-1} \overset{s,t}{\rightrightarrows} \dots \overset{s,t}{\rightrightarrows} L_0 \rightrightarrows 0, \tag{3.1}$$

of vector spaces L_m and linear maps s, t such that

$$s(s(a)) = s(t(a)) \text{ and } t(s(a)) = t(t(a)), \tag{3.2}$$

for any $a \in L_m$, $m \in \{1, \dots, n\}$. The maps s, t are called **source map** and **target map**, respectively, and any element of L_m is an **m-cell**.

By higher category we mean in this text a **strict** higher category. Roughly, a linear n-category, $n \in \mathbb{N}$, is an n-globular vector space endowed with compositions of m-cells, $0 < m \leq n$, along a p-cell, $0 \leq p < m$, and an identity associated to any m-cell, $0 \leq m < n$. Two m-cells $(a, b) \in L_m \times L_m$ are composable along a p-cell, if $t^{m-p}(a) = s^{m-p}(b)$. The composite m-cell will be denoted by $a \circ_p b$ (the cell that 'acts' first is written on the left) and the vector subspace of $L_m \times L_m$ made up by the pairs of m-cells that can be composed along a p-cell will be denoted by $L_m \times_{L_p} L_m$. The following figure schematizes the composition of two 3-cells along a 0-, a 1-, and a 2-cell.



Definition 3.2.3. A **linear n-category**, $n \in \mathbb{N}$, is an n-globular vector space L (with source and target maps s, t) together with, for any $m \in \{1, \ldots, n\}$ and any $p \in \{0, \ldots, m-1\}$, a linear composition map $\circ_p : L_m \times_{L_p} L_m \to L_m$ and, for any $m \in \{0, \ldots, n-1\}$, a linear identity map $1 : L_m \to L_{m+1}$, such that the properties

- for $(a, b) \in L_m \times_{L_p} L_m$,
$$\text{if } \ p = m-1, \text{ then } \ s(a \circ_p b) = s(a) \text{ and } \ t(a \circ_p b) = t(b),$$
$$\text{if } \ p \leq m-2, \text{ then } \ s(a \circ_p b) = s(a) \circ_p s(b) \text{ and } \ t(a \circ_p b) = t(a) \circ_p t(b),$$
- $$s(1_a) = t(1_a) = a,$$
- for any $(a, b), (b, c) \in L_m \times_{L_p} L_m$,
$$(a \circ_p b) \circ_p c = a \circ_p (b \circ_p c),$$
- $$1^{m-p}_{s^{m-p}a} \circ_p a = a \circ_p 1^{m-p}_{t^{m-p}a} = a$$

are verified, as well as the compatibility conditions

- for $q < p$, $(a, b), (c, d) \in L_m \times_{L_p} L_m$ and $(a, c), (b, d) \in L_m \times_{L_q} L_m$,
$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d),$$
- for $m < n$ and $(a, b) \in L_m \times_{L_p} L_m$,
$$1_{a \circ_p b} = 1_a \circ_p 1_b.$$

The morphisms between two linear n-categories are the linear n-functors.

Definition 3.2.4. A **linear n-functor** $F : L \to L'$ between two linear n-categories is made up by linear maps $F : L_m \to L'_m$, $m \in \{0, \ldots, n\}$, such that the categorical structure – source and target maps, composition maps, identity maps – is respected.

Linear n-categories and linear n-functors form a category `Vect` n-`Cat`, see Proposition 3.2.1. To disambiguate this proposition, let us specify that the objects of $\mathtt{C}^{n+1}(\mathtt{Vect})$ are the complexes whose underlying vector space $V = \oplus_{i=0}^{n} V_i$ is made up by $n+1$ terms V_i.

The proof of Proposition 3.2.1 is based upon the following result.

Proposition 3.2.5. *Let L be any n-globular vector space with linear identity maps. If s_m denotes the restriction of the source map to L_m, the vector spaces L_m and $L'_m := \oplus_{i=0}^{m} V_i$, $V_i := \ker s_i$, $m \in \{0, \ldots, n\}$, are isomorphic. Further, the n-globular vector space with identities can be completed in a unique way by linear composition maps so to form a linear n-category. If we identify L_m with L'_m, this unique linear n-categorical structure reads*

$$s(v_0, \ldots, v_m) = (v_0, \ldots, v_{m-1}), \tag{3.3}$$

$$t(v_0, \ldots, v_m) = (v_0, \ldots, v_{m-1} + tv_m), \tag{3.4}$$

$$1_{(v_0,\ldots,v_m)} = (v_0, \ldots, v_m, 0), \tag{3.5}$$

$$(v_0, \ldots, v_m) \circ_p (v'_0, \ldots, v'_m) = (v_0, \ldots, v_p, v_{p+1} + v'_{p+1}, \ldots, v_m + v'_m), \tag{3.6}$$

where the two m-cells in Equation (3.6) are assumed to be composable along a p-cell.

Proof. As for the first part of this proposition, if $m = 2$ e.g., it suffices to observe that the linear maps

$$\alpha_L : L'_2 = V_0 \oplus V_1 \oplus V_2 \ni (v_0, v_1, v_2) \mapsto 1^2_{v_0} + 1_{v_1} + v_2 \in L_2$$

and

$$\beta_L : L_2 \ni a \mapsto (s^2 a, s(a - 1^2_{s^2 a}), a - 1_{s(a - 1^2_{s^2 a})} - 1^2_{s^2 a}) \in V_0 \oplus V_1 \oplus V_2 = L'_2$$

are inverses of each other. For arbitrary $m \in \{0, \ldots, n\}$ and $a \in L_m$, we set

$$\beta_L a = \left(s^m a, \ldots, s^{m-i}(a - \sum_{j=0}^{i-1} 1^{m-j}_{p_j \beta_L a}), \ldots, a - \sum_{j=0}^{m-1} 1^{m-j}_{p_j \beta_L a} \right) \in V_0 \oplus \ldots \oplus V_i \oplus \ldots \oplus V_m = L'_m,$$

where p_j denotes the projection $p_j : L'_m \to V_j$ and where the components must be computed from left to right.

For the second claim, note that when reading the source, target and identity maps through the detailed isomorphism, we get $s(v_0, \ldots, v_m) = (v_0, \ldots, v_{m-1})$, $t(v_0, \ldots, v_m) = (v_0, \ldots, v_{m-1} + tv_m)$, and $1_{(v_0,\ldots,v_m)} = (v_0, \ldots, v_m, 0)$. Eventually, set $v = (v_0, \ldots, v_m)$ and let (v, w) and (v', w') be two pairs of m-cells that are composable along a p-cell. The composability condition, say for (v, w), reads

$$(w_0, \ldots, w_p) = (v_0, \ldots, v_{p-1}, v_p + tv_{p+1}).$$

It follows from the linearity of $\circ_p : L_m \times_{L_p} L_m \to L_m$ that $(v+v') \circ_p (w+w') = (v \circ_p w) + (v' \circ_p w')$. When taking $w = 1^{m-p}_{t^{m-p}v}$ and $v' = 1^{m-p}_{s^{m-p}w'}$, we find

$$(v_0 + w'_0, \ldots, v_p + w'_p, v_{p+1}, \ldots, v_m) \circ_p (v_0 + w'_0, \ldots, v_p + w'_p + tv_{p+1}, w'_{p+1}, \ldots, w'_m)$$
$$= (v_0 + w'_0, \ldots, v_m + w'_m),$$

so that $\circ_p$ is necessarily the composition given by Equation (3.6). It is easily seen that, conversely, Equations (3.3) – (3.6) define a linear n-categorical structure. □

Proof of Proposition 3.2.1. We define functors $\mathfrak{N} : \texttt{Vect}\ n\text{-}\texttt{Cat} \to \texttt{C}^{n+1}(\texttt{Vect})$ and $\mathfrak{G} : \texttt{C}^{n+1}(\texttt{Vect}) \to \texttt{Vect}\ n\text{-}\texttt{Cat}$ that are inverses up to natural isomorphisms.

If we start from a linear n-category L, so in particular from an n-globular vector space L, we define an $(n+1)$-term chain complex $\mathfrak{N}(L)$ by setting $V_m = \ker s_m \subset L_m$ and $d_m = t_m|_{V_m} : V_m \to V_{m-1}$. In view of the globular space conditions (3.2), the target space of d_m is actually V_{m-1} and we have $d_{m-1} d_m v_m = 0$.

Moreover, if $F : L \to L'$ denotes a linear n-functor, the value $\mathfrak{N}(F) : V \to V'$ is defined on $V_m \subset L_m$ by $\mathfrak{N}(F)_m = F_m|_{V_m} : V_m \to V'_m$. It is obvious that $\mathfrak{N}(F)$ is a linear chain map.

It is obvious that $\mathfrak{N}$ respects the categorical structures of $\texttt{Vect}\ n\text{-}\texttt{Cat}$ and $\texttt{C}^{n+1}(\texttt{Vect})$.

As for the second functor $\mathfrak{G}$, if (V, d), $V = \oplus_{i=0}^n V_i$, is an $(n+1)$-term chain complex of vector spaces, we define a linear n-category $\mathfrak{G}(V) = L$, $L_m = \oplus_{i=0}^m V_i$, as in Proposition 3.2.5: the source, target, identity and composition maps are defined by Equations (3.3) – (3.6), except that tv_m in the RHS of Equation (3.4) is replaced by dv_m.

The definition of $\mathfrak{G}$ on a linear chain map $\phi : V \to V'$ leads to a linear n-functor $\mathfrak{G}(\phi) : L \to L'$, which is defined on $L_m = \oplus_{i=0}^m V_i$ by $\mathfrak{G}(\phi)_m = \oplus_{i=0}^m \phi_i$. Indeed, it is readily checked that $\mathfrak{G}(\phi)$ respects the linear n-categorical structures of L and L'.

Furthermore, $\mathfrak{G}$ respects the categorical structures of $\texttt{C}^{n+1}(\texttt{Vect})$ and $\texttt{Vect}\ n\text{-}\texttt{Cat}$.

Eventually, there exists natural isomorphisms $\alpha : \mathfrak{N}\mathfrak{G} \Rightarrow \mathrm{id}$ and $\gamma : \mathfrak{G}\mathfrak{N} \Rightarrow \mathrm{id}$.

To define a natural transformation $\alpha : \mathfrak{N}\mathfrak{G} \Rightarrow \mathrm{id}$, note that $L' = (\mathfrak{N}\mathfrak{G})(L)$ is the linear n-category made up by the vector spaces $L'_m = \oplus_{i=0}^m V_i$, $V_i = \ker s_i$, as well as by the source, target, identities and compositions defined from $V = \mathfrak{N}(L)$ as in the above definition of $\mathfrak{G}(V)$, i.e. as in Proposition 3.2.5. It follows that $\alpha_L : L' \to L$, defined by $\alpha_L : L'_m \ni (v_0, \ldots, v_m) \mapsto 1^m_{v_0} + \ldots + 1_{v_{m-1}} + v_m \in L_m$, $m \in \{0, \ldots, n\}$, which pulls the linear n-categorical structure back from L to L', see Proposition 3.2.5, is an invertible linear n-functor. Moreover α is natural in L.

It suffices now to observe that the composite $\mathfrak{G}\mathfrak{N}$ is the identity functor. □

Next we further investigate the category $\texttt{Vect}\ n\text{-}\texttt{Cat}$.

Proposition 3.2.6. *The category* $\texttt{Vect}\ n\text{-}\texttt{Cat}$ *admits finite products.*

Let L and L' be two linear n-categories. The product linear n-category $L \times L'$ is defined by $(L \times L')_m = L_m \times L'_m$, $S_m = s_m \times s'_m$, $T_m = t_m \times t'_m$, $I_m = 1_m \times 1'_m$, and $\bigcirc_p = \circ_p \times \circ'_p$. The compositions $\bigcirc_p$ coincide with the unique compositions that complete the n-globular vector space with identities, thus providing a linear n-category. It is straightforwardly checked that the product of linear n-categories verifies the universal property for binary products.

Proposition 3.2.7. *The category* `Vect 2-Cat` *admits a 3-categorical structure. More precisely, its 2-cells are the linear natural 2-transformations and its 3-cells are the linear 2-modifications.*

This proposition is the linear version (with similar proof) of the well-known result that the category `2-Cat` is a 3-category with 2-categories as 0-cells, 2-functors as 1-cells, natural 2-transformations as 2-cells, and 2-modifications as 3-cells. The definitions of n-categories and 2-functors are similar to those given above in the linear context (but they are formulated without the use of set theoretical concepts). As for (linear) natural 2-transformations and (linear) 2-modifications, let us recall their definition in the linear setting:

Definition 3.2.8. A **linear natural 2-transformation** $\theta : F \Rightarrow G$ between two linear 2-functors $F, G : \mathcal{C} \to \mathcal{D}$, between the same two linear 2-categories, assigns to any $a \in \mathcal{C}_0$ a unique $\theta_a : F(a) \to G(a)$ in $\mathcal{D}_1$, linear with respect to a and such that for any $\alpha : f \Rightarrow g$ in $\mathcal{C}_2$, $f, g : a \to b$ in $\mathcal{C}_1$, we have

$$F(\alpha) \circ_0 1_{\theta_b} = 1_{\theta_a} \circ_0 G(\alpha) . \tag{3.7}$$

If $\mathcal{C} = L \times L$ is a product linear 2-category, the last condition reads

$$F(\alpha, \beta) \circ_0 1_{\theta_{t^2\alpha, t^2\beta}} = 1_{\theta_{s^2\alpha, s^2\beta}} \circ_0 G(\alpha, \beta),$$

for all $(\alpha, \beta) \in L_2 \times L_2$. As functors respect composition, i.e. as

$$F(\alpha, \beta) = F(\alpha \circ_0 1^2_{t^2\alpha}, 1^2_{s^2\beta} \circ_0 \beta) = F(\alpha, 1^2_{s^2\beta}) \circ_0 F(1^2_{t^2\alpha}, \beta),$$

this naturality condition is verified if and only if it holds true in case all but one of the 2-cells are identities 1^2_{-}, i.e. if and only if the transformation is natural with respect to all its arguments separately.

Definition 3.2.9. Let $\mathcal{C}, \mathcal{D}$ be two linear 2-categories. A **linear 2-modification** $\mu : \eta \Rrightarrow \varepsilon$ between two linear natural 2-transformations $\eta, \varepsilon : F \Rightarrow G$, between the same two linear 2-functors $F, G : \mathcal{C} \to \mathcal{D}$, assigns to any object $a \in \mathcal{C}_0$ a unique $\mu_a : \eta_a \Rightarrow \varepsilon_a$ in $\mathcal{D}_2$, which is linear with respect to a and such that, for any $\alpha : f \Rightarrow g$ in $\mathcal{C}_2$, $f, g : a \to b$ in $\mathcal{C}_1$, we have

$$F(\alpha) \circ_0 \mu_b = \mu_a \circ_0 G(\alpha). \tag{3.8}$$

If $\mathcal{C} = L \times L$ is a product linear 2-category, it suffices again that the preceding modification property be satisfied for tuples (α, β), in which all but one 2-cells are identities 1^2_{-}. The explanation is the same as for natural transformations.

Beyond linear 2-functors, linear natural 2-transformations, and linear 2-modifications, we use below multilinear cells. Bilinear cells e.g., are cells on a product linear 2-category, with linearity replaced by bilinearity. For instance,

Definition 3.2.10. Let L, L', and L'' be linear 2-categories. A **bilinear 2-functor** $F : L \times L' \to L''$ is a 2-functor such that $F : L_m \times L'_m \to L''_m$ is bilinear for all $m \in \{0, 1, 2\}$.

Similarly,

Definition 3.2.11. Let L, L', and L'' be linear 2-categories. A **bilinear natural 2-transformation** $\theta : F \Rightarrow G$ between two bilinear 2-functors $F, G : L \times L' \to L''$, assigns to any $(a, b) \in L_0 \times L'_0$ a unique $\theta_{(a,b)} : F(a, b) \to G(a, b)$ in L''_1, which is bilinear with respect to (a, b) and such that for any $(\alpha, \beta) : (f, h) \Rightarrow (g, k)$ in $L_2 \times L'_2$, $(f, h), (g, k) : (a, b) \to (c, d)$ in $L_1 \times L'_1$, we have

$$F(\alpha, \beta) \circ_0 1_{\theta_{(c,d)}} = 1_{\theta_{(a,b)}} \circ_0 G(\alpha, \beta) . \tag{3.9}$$

3.3 Homotopy Lie algebras and categorified Lie algebras

We now recall the definition of a Lie infinity (strongly homotopy Lie, sh Lie, $L_\infty-$) algebra and specify it in the case of a 3-term Lie infinity algebra.

Definition 3.3.1. A **Lie infinity algebra** is an N-graded vector space $V = \oplus_{i\in\mathrm{N}} V_i$ together with a family $(\ell_i)_{i\in\mathrm{N}^*}$ of graded antisymmetric i-linear weight $i-2$ maps on V, which verify the sequence of conditions

$$\sum_{i+j=n+1} \sum_{(i,n-i)\,-\,\text{shuffles }\sigma} \chi(\sigma)(-1)^{i(j-1)} \ell_j(\ell_i(a_{\sigma_1}, \ldots, a_{\sigma_i}), a_{\sigma_{i+1}}, \ldots, a_{\sigma_n}) = 0, \tag{3.10}$$

where $n \in \{1, 2, \ldots\}$, where $\chi(\sigma)$ is the product of the signature of σ and the Koszul sign defined by σ and the homogeneous arguments $a_1, \ldots, a_n \in V$.

For $n = 1$, the L_∞-condition (3.10) reads $\ell_1^2 = 0$ and, for $n = 2$, it means that ℓ_1 is a graded derivation of ℓ_2, or, equivalently, that ℓ_2 is a chain map from $(V \otimes V, \ell_1 \otimes \mathrm{id} + \mathrm{id} \otimes\ \ell_1)$ to (V, ℓ_1).

In particular,

Definition 3.3.2. A **3-term Lie infinity algebra** is a 3-term graded vector space $V = V_0 \oplus V_1 \oplus V_2$ endowed with graded antisymmetric p-linear maps ℓ_p of weight $p - 2$,

$$\begin{array}{ll} \ell_1 : V_i \to V_{i-1} & (1 \le i \le 2), \\ \ell_2 : V_i \times V_j \to V_{i+j} & (0 \le i + j \le 2), \\ \ell_3 : V_i \times V_j \times V_k \to V_{i+j+k+1} & (0 \le i + j + k \le 1), \\ \ell_4 : V_0 \times V_0 \times V_0 \times V_0 \to V_2 & \end{array} \tag{3.11}$$

(all structure maps ℓ_p, $p > 4$, necessarily vanish), which satisfy L_∞-condition (3.10) (that is trivial for all $n > 5$).

In this 3-term situation, each L_∞-condition splits into a finite number of equations determined by the various combinations of argument degrees, see below.

On the other hand, we have the

Definition 3.3.3. A **Lie 3-algebra** is a **linear 2-category** L endowed with a **bracket**, i.e. an antisymmetric bilinear 2-functor $[-,-] : L \times L \to L$, which verifies the Jacobi identity up to a **Jacobiator**, i.e. a skew-symmetric trilinear natural 2-transformation

$$J_{xyz} : [[x, y], z] \to [[x, z], y] + [x, [y, z]], \tag{3.12}$$

$x, y, z \in L_0$, which in turn satisfies the Baez-Crans Jacobiator identity up to an **Identiator**, i.e. a skew-symmetric quadrilinear 2-modification

$$\begin{aligned} \mu_{xyzu} : &[J_{x,y,z}, 1_u] \circ_0 (J_{[x,z],y,u} + J_{x,[y,z],u}) \circ_0 ([J_{xzu}, 1_y] + 1) \circ_0 ([1_x, J_{yzu}] + 1) \\ &\Rightarrow J_{[x,y],z,u} \circ_0 ([J_{xyu}, 1_z] + 1) \circ_0 (J_{x,[y,u],z} + J_{[x,u],y,z} + J_{x,y,[z,u]}), \end{aligned} \tag{3.13}$$

$x, y, z, u \in L_0$, required to verify the **coherence law**

$$\alpha_1 + \alpha_4^{-1} = \alpha_3 + \alpha_2^{-1}, \tag{3.14}$$

where $\alpha_1 - \alpha_4$ are explicitly given in Definitions 3.4.16 – 3.4.19 and where superscript -1 denotes the inverse for composition along a 1-cell.

Just as the *Jacobiator* is a natural transformation between the two sides of the *Jacobi* identity, the *Identiator* is a modification between the two sides of the Baez-Crans Jacobiator *identity*.

In this definition "skew-symmetric 2-transformation" (resp. "skew-symmetric 2-modification") means that, if we identify L_m with $\oplus_{i=0}^m V_i$, $V_i = \ker s_i$, as in Proposition 3.2.5, the V_1-component of $J_{xyz} \in L_1$ (resp. the V_2-component of $\mu_{xyzu} \in L_2$) is antisymmetric. Moreover, the definition makes sense, as the source and target in Equation (3.13) are quadrilinear natural 2-transformations between quadrilinear 2-functors from $L^{\times 4}$ to L. These 2-functors are simplest obtained from the RHS of Equation (3.13). Further, the mentioned source and target actually are natural 2-transformations, since a 2-functor composed (on the left or on the right) with a natural 2-transformation is again a 2-transformation.

3.4 Lie 3-algebras in comparison with 3-term Lie infinity algebras

Remark 3.4.1. In the following, we systematically identify the vector spaces L_m, $m \in \{0, \ldots, n\}$, of a linear n-category with the spaces $L'_m = \oplus_{i=0}^m V_i$, $V_i = \ker s_i$, so that the categorical structure is given by Equations (3.3) – (3.6). In addition, we often substitute common, index-free notations (e.g. $\alpha = (x, \mathbf{f}, \mathbf{a})$) for our notations (e.g. $v = (v_0, v_1, v_2) \in L_2$).

The next theorem is the main result of this paper.

Theorem 3.4.2. *There exists a 1-to-1 correspondence between Lie 3-algebras and 3-term Lie infinity algebras (V, ℓ_p), whose structure maps ℓ_2 and ℓ_3 vanish on $V_1 \times V_1$ and on triplets of total degree 1, respectively.*

Example 3.4.3. There exists a 1-to-1 correspondence between $(n+1)$-term Lie infinity algebras $V = V_0 \oplus V_n$ (whose intermediate terms vanish), $n \geq 2$, and $(n+2)$-cocycles of Lie algebras endowed with a linear representation, see [BC04], Theorem 6.7. A 3-term Lie infinity algebra implemented by a 4-cocycle can therefore be viewed as a special case of a Lie 3-algebra.

The proof of Theorem 3.4.2 consists of five lemmas.

3.4.1 Linear 2-category – three term chain complex of vector spaces

First, we recall the correspondence between the underlying structures of a Lie 3-algebra and a 3-term Lie infinity algebra.

Lemma 3.4.4. *There is a bijective correspondence between linear 2-categories L and 3-term chain complexes of vector spaces (V, ℓ_1).*

Proof. In the proof of Proposition 3.2.1, we associated to any linear 2-category L a unique 3-term chain complex of vector spaces $\mathfrak{N}(L) = V$, whose spaces are given by $V_m = \ker s_m$, $m \in \{0, 1, 2\}$, and whose differential ℓ_1 coincides on V_m with the restriction $t_m|_{V_m}$. Conversely, we assigned to any such chain complex V a unique linear 2-category $\mathfrak{G}(V) = L$, with spaces $L_m = \oplus_{i=0}^m V_i$, $m \in \{0, 1, 2\}$ and target $t_0(x) = 0$, $t_1(x, \mathbf{f}) = x + \ell_1 \mathbf{f}$, $t_2(x, \mathbf{f}, \mathbf{a}) = (x, \mathbf{f} + \ell_1 \mathbf{a})$. In view of Remark 3.4.1, the maps $\mathfrak{N}$ and $\mathfrak{G}$ are inverses of each other. □

Remark 3.4.5. The globular space condition is the categorical counterpart of L_∞-condition $n = 1$.

3.4.2 Bracket – chain map

We assume that we already built (V, ℓ_1) from L or L from (V, ℓ_1).

Lemma 3.4.6. *There is a bijective correspondence between antisymmetric bilinear 2-functors* $[-,-]$ *on* L *and graded antisymmetric chain maps* $\ell_2 : (V \otimes V, \ell_1 \otimes \mathrm{id} + \mathrm{id} \otimes \ell_1) \to (V, \ell_1)$ *that vanish on* $V_1 \times V_1$.

Proof. Consider first an antisymmetric bilinear "2-map" $[-,-] : L \times L \to L$ that verifies all functorial requirements except as concerns composition. This bracket then respects the compositions, i.e., for each pairs $(v, w), (v', w') \in L_m \times L_m$, $m \in \{1, 2\}$, that are composable along a p-cell, $0 \le p < m$, we have

$$[v \circ_p v', w \circ_p w'] = [v, w] \circ_p [v', w'], \tag{3.15}$$

if and only if the following conditions hold true, for any $\mathbf{f}, \mathbf{g} \in V_1$ and any $\mathbf{a}, \mathbf{b} \in V_2$:

$$[\mathbf{f}, \mathbf{g}] = [1_{t\mathbf{f}}, \mathbf{g}] = [\mathbf{f}, 1_{t\mathbf{g}}], \tag{3.16}$$

$$[\mathbf{a}, \mathbf{b}] = [1_{t\mathbf{a}}, \mathbf{b}] = [\mathbf{a}, 1_{t\mathbf{b}}] = 0, \tag{3.17}$$

$$[1_{\mathbf{f}}, \mathbf{b}] = [1^2_{t\mathbf{f}}, \mathbf{b}] = 0. \tag{3.18}$$

To prove the first two conditions, it suffices to compute $[\mathbf{f} \circ_0 1_{t\mathbf{f}}, 1_0 \circ_0 \mathbf{g}]$, for the next three conditions, we consider $[\mathbf{a} \circ_1 1_{t\mathbf{a}}, 1_0 \circ_1 \mathbf{b}]$ and $[\mathbf{a} \circ_0 1^2_0, 1^2_0 \circ_0 \mathbf{b}]$, and for the last two, we focus on $[1_{\mathbf{f}} \circ_0 1^2_{t\mathbf{f}}, 1^2_0 \circ_0 \mathbf{b}]$ and $[1_{\mathbf{f}} \circ_0 (1^2_{t\mathbf{f}} + 1_{\mathbf{f}'}), \mathbf{b} \circ_0 \mathbf{b}']$. Conversely, it can be straightforwardly checked that Equations (3.16) – (3.18) entail the general requirement (3.15).

On the other hand, a graded antisymmetric bilinear weight 0 map $\ell_2 : V \times V \to V$ commutes with the differentials ℓ_1 and $\ell_1 \otimes \mathrm{id} + \mathrm{id} \otimes \ell_1$, i.e., for all $v, w \in V$, we have

$$\ell_1(\ell_2(v, w)) = \ell_2(\ell_1 v, w) + (-1)^v \ell_2(v, \ell_1 w) \tag{3.19}$$

(we assumed that v is homogeneous and denoted its degree by v as well), if and only if, for any $y \in V_0$, $\mathbf{f}, \mathbf{g} \in V_1$, and $\mathbf{a} \in V_2$,

$$\ell_1(\ell_2(\mathbf{f}, y)) = \ell_2(\ell_1 \mathbf{f}, y), \tag{3.20}$$

$$\ell_1(\ell_2(\mathbf{f}, \mathbf{g})) = \ell_2(\ell_1 \mathbf{f}, \mathbf{g}) - \ell_2(\mathbf{f}, \ell_1 \mathbf{g}), \tag{3.21}$$

$$\ell_1(\ell_2(\mathbf{a}, y)) = \ell_2(\ell_1 \mathbf{a}, y), \tag{3.22}$$

$$0 = \ell_2(\ell_1 \mathbf{f}, \mathbf{b}) - \ell_2(\mathbf{f}, \ell_1 \mathbf{b}). \tag{3.23}$$

Remark 3.4.7. Note that, in the correspondence $\ell_1 \leftrightarrow t$ and $\ell_2 \leftrightarrow [-,-]$, Equations (3.20) and (3.22) read as compatibility requirements of the bracket with the target and that Equations (3.21) and (3.23) correspond to the second conditions of Equations (3.16) and (3.18), respectively.

Proof of Lemma 3.4.6 (continuation). To prove the announced 1-to-1 correspondence, we first define a graded antisymmetric chain map $\mathfrak{N}([-,-]) = \ell_2$, $\ell_2 : V \otimes V \to V$ from any antisymmetric bilinear 2-functor $[-,-] : L \times L \to L$.

Let $x, y \in V_0$, $\mathbf{f}, \mathbf{g} \in V_1$, and $\mathbf{a}, \mathbf{b} \in V_2$. Set $\ell_2(x, y) = [x, y] \in V_0$ and $\ell_2(x, \mathbf{g}) = [1_x, \mathbf{g}] \in V_1$. However, we must define $\ell_2(\mathbf{f}, \mathbf{g}) \in V_2$, whereas $[\mathbf{f}, \mathbf{g}] \in V_1$. Moreover, in this case, the antisymmetry properties do not match. The observation

$$[\mathbf{f}, \mathbf{g}] = [1_{t\mathbf{f}}, \mathbf{g}] = [\mathbf{f}, 1_{t\mathbf{g}}] = \ell_2(\ell_1 \mathbf{f}, \mathbf{g}) = \ell_2(\mathbf{f}, \ell_1 \mathbf{g})$$

and Condition (3.21) force us to *define* ℓ_2 *on* $V_1 \times V_1$ *as a symmetric bilinear map valued in* $V_2 \cap \ker \ell_1$. We further set $\ell_2(x, \mathbf{b}) = [1_x^2, \mathbf{b}] \in V_2$, and, as ℓ_2 is required to have weight 0, we must set $\ell_2(\mathbf{f}, \mathbf{b}) = 0$ and $\ell_2(\mathbf{a}, \mathbf{b}) = 0$. It then follows from the functorial properties of $[-,-]$ that the conditions (3.20) – (3.22) are verified. In view of Equation (3.18), Property (3.23) reads

$$0 = [1_{t\mathbf{f}}^2, \mathbf{b}] - \ell_2(\mathbf{f}, \ell_1\mathbf{b}) = -\ell_2(\mathbf{f}, \ell_1\mathbf{b}).$$

In other words, in addition to the preceding requirement, we must *choose* ℓ_2 *in a way that it vanishes on* $V_1 \times V_1$ *if evaluated on a 1-coboundary.* These conditions are satisfied if we choose $\ell_2 = 0$ on $V_1 \times V_1$.

Conversely, from any graded antisymmetric chain map ℓ_2 that vanishes on $V_1 \times V_1$, we can construct an antisymmetric bilinear 2-functor $\mathfrak{G}(\ell_2) = [-,-]$. Indeed, using obvious notations, we set

$$[x, y] = \ell_2(x, y) \in L_0,\ [1_x, 1_y] = 1_{[x,y]} \in L_1,\ [1_x, \mathbf{g}] = \ell_2(x, \mathbf{g}) \in V_1 \subset L_1.$$

Again $[\mathbf{f}, \mathbf{g}] \in L_1$ cannot be defined as $\ell_2(\mathbf{f}, \mathbf{g}) \in V_2$. Instead, if we wish to build a 2-functor, we must set

$$[\mathbf{f}, \mathbf{g}] = [1_{t\mathbf{f}}, \mathbf{g}] = [\mathbf{f}, 1_{t\mathbf{g}}] = \ell_2(\ell_1\mathbf{f}, \mathbf{g}) = \ell_2(\mathbf{f}, \ell_1\mathbf{g}) \in V_1 \subset L_1,$$

which is possible in view of Equation (3.21), *if* ℓ_2 *is on* $V_1 \times V_1$ *valued in 2-cocycles* (and in particular if it vanishes on this subspace). Further, we define

$$[1_x^2, 1_y^2] = 1_{[x,y]}^2 \in L_2,\ [1_x^2, 1_\mathbf{g}] = 1_{[1_x,\mathbf{g}]} \in L_2,\ [1_x^2, \mathbf{b}] = \ell_2(x, \mathbf{b}) \in V_2 \subset L_2,\ [1_\mathbf{f}, 1_\mathbf{g}] = 1_{[\mathbf{f},\mathbf{g}]} \in L_2.$$

Finally, we must set

$$[1_\mathbf{f}, \mathbf{b}] = [1_{t\mathbf{f}}^2, \mathbf{b}] = \ell_2(\ell_1\mathbf{f}, \mathbf{b}) = 0,$$

which is possible in view of Equation (3.23), *if* ℓ_2 *vanishes on* $V_1 \times V_1$ *when evaluated on a 1-coboundary* (and especially if it vanishes on the whole subspace $V_1 \times V_1$), and

$$[\mathbf{a}, \mathbf{b}] = [1_{t\mathbf{a}}, \mathbf{b}] = [\mathbf{a}, 1_{t\mathbf{b}}] = 0,$$

which is possible.

It follows from these definitions that the bracket of $\alpha = (x, \mathbf{f}, \mathbf{a}) = 1_x^2 + 1_\mathbf{f} + \mathbf{a} \in L_2$ and $\beta = (y, \mathbf{g}, \mathbf{b}) = 1_y^2 + 1_\mathbf{g} + \mathbf{b} \in L_2$ is given by

$$[\alpha, \beta] = (\ell_2(x, y), \ell_2(x, \mathbf{g}) + \ell_2(\mathbf{f}, tg), \ell_2(x, \mathbf{b}) + \ell_2(\mathbf{a}, y)) \in L_2, \tag{3.24}$$

where $g = (y, \mathbf{g})$. The brackets of two elements of L_1 or L_0 are obtained as special cases of the latter result.

We thus defined an antisymmetric bilinear map $[-,-]$ that assigns an i-cell to any pair of i-cells, $i \in \{0, 1, 2\}$, and that respects identities and sources. Moreover, since Equations (3.16) – (3.18) are satisfied, the map $[-,-]$ respects compositions provided it respects targets. For the last of the first three defined brackets, the target condition is verified due to Equation (3.20). For the fourth bracket, the target must coincide with $[t\mathbf{f}, t\mathbf{g}] = \ell_2(\ell_1\mathbf{f}, \ell_1\mathbf{g})$ and it actually coincides with $t[\mathbf{f}, \mathbf{g}] = \ell_1\ell_2(\ell_1\mathbf{f}, \mathbf{g}) = \ell_2(\ell_1\mathbf{f}, \ell_1\mathbf{g})$, again in view of (3.20). As regards the seventh bracket, the target $t[1_x^2, \mathbf{b}] = \ell_1\ell_2(x, \mathbf{b}) = \ell_2(x, \ell_1\mathbf{b})$, due to (3.22), must coincide with $[1_x, t\mathbf{b}] = \ell_2(x, \ell_1\mathbf{b})$. The targets of the two last brackets vanish and $[\mathbf{f}, t\mathbf{b}] = \ell_2(\mathbf{f}, \ell_1\ell_1\mathbf{b}) = 0$ and $[t\mathbf{a}, t\mathbf{b}] = \ell_2(\ell_1\mathbf{a}, \ell_1\ell_1\mathbf{b}) = 0$.

It is straightforwardly checked that the maps $\mathfrak{N}$ and $\mathfrak{G}$ are inverses. $\square$

Note that $\mathfrak{N}$ actually assigns to any antisymmetric bilinear 2-functor a class of graded antisymmetric chain maps that coincide outside $V_1 \times V_1$ and whose restrictions to $V_1 \times V_1$ are valued in 2-cocycles and vanish when evaluated on a 1-coboundary. The map $\mathfrak{N}$, with values in chain maps, is well-defined thanks to a canonical choice of a representative of this class. Conversely, the values on $V_1 \times V_1$ of the considered chain map cannot be encrypted into the associated 2-functor, only the mentioned cohomological conditions are of importance. Without the canonical choice, the map $\mathfrak{G}$ would not be injective.

Remark 3.4.8. The categorical counterpart of L_∞-condition $n = 2$ is the functor condition on compositions.

Remark 3.4.9. A 2-term Lie infinity algebra (resp. a Lie 2-algebra) can be viewed as a 3-term Lie infinity algebra (resp. a Lie 3-algebra). The preceding correspondence then of course reduces to the correspondence of [BC04].

3.4.3 Jacobiator – third structure map

We suppose that we already constructed (V, ℓ_1, ℓ_2) from $(L, [-,-])$ or $(L, [-,-])$ from (V, ℓ_1, ℓ_2).

Lemma 3.4.10. *There exists a bijective correspondence between skew-symmetric trilinear natural 2-transformations* $J : [[-,-],\bullet] \Rightarrow [[-,\bullet],-] + [-,[-,\bullet]]$ *and graded antisymmetric trilinear weight 1 maps* $\ell_3 : V^{\times 3} \to V$ *that verify* L_∞*-condition* $n = 3$ *and vanish in total degree 1.*

Proof. A skew-symmetric trilinear natural 2-transformation $J : [[-,-],\bullet] \Rightarrow [[-,\bullet],-] + [-,[-,\bullet]]$ is a map that assigns to any $(x,y,z) \in L_0^{\times 3}$ a unique $J_{xyz} : [[x,y],z] \to [[x,z],y] + [x,[y,z]]$ in L_1, such that for any $\alpha = (z, \mathbf{f}, \mathbf{a}) \in L_2$, we have

$$[[1_x^2, 1_y^2], \alpha] \circ_0 1_{J_{x,y,t^2\alpha}} = 1_{J_{x,y,s^2\alpha}} \circ_0 \left([[1_x^2, \alpha], 1_y^2] + [1_x^2, [1_y^2, \alpha]]\right)$$

(as well as similar equations pertaining to naturality with respect to the other two variables). A short computation shows that the last condition decomposes into the following two requirements on the V_1- and the V_2-component:

$$\mathbf{J}_{x,y,t\mathbf{f}} + [1_{[x,y]}, \mathbf{f}] = [[1_x, \mathbf{f}], 1_y] + [1_x, [1_y, \mathbf{f}]], \tag{3.25}$$

$$[1^2_{[x,y]}, \mathbf{a}] = [[1_x^2, \mathbf{a}], 1_y^2] + [1_x^2, [1_y^2, \mathbf{a}]]. \tag{3.26}$$

A graded antisymmetric trilinear weight 1 map $\ell_3 : V^{\times 3} \to V$ verifies L_∞-condition $n = 3$ if

$$\ell_1(\ell_3(u,v,w)) + \ell_2(\ell_2(u,v),w) - (-1)^{vw}\ell_2(\ell_2(u,w),v) + (-1)^{u(v+w)}\ell_2(\ell_2(v,w),u) +$$
$$+\ell_3(\ell_1(u),v,w) - (-1)^{uv}\ell_3(\ell_1(v),u,w) + (-1)^{w(u+v)}\ell_3(\ell_1(w),u,v) = 0, \tag{3.27}$$

for any homogeneous $u, v, w \in V$. This condition is trivial for any arguments of total degree $d = u + v + w > 2$. For $d = 0$, we write $(u,v,w) = (x,y,z) \in V_0^{\times 3}$, for $d = 1$, we consider $(u,v) = (x,y) \in V_0^{\times 2}$ and $w = \mathbf{f} \in V_1$, for $d = 2$, either $(u,v) = (x,y) \in V_0^{\times 2}$ and $w = \mathbf{a} \in V_2$, or $u = x \in V_0$ and $(v,w) = (\mathbf{f}, \mathbf{g}) \in V_1^{\times 2}$, so that Equation (3.27) reads

$$\ell_1(\ell_3(x,y,z)) + \ell_2(\ell_2(x,y),z) - \ell_2(\ell_2(x,z),y) + \ell_2(\ell_2(y,z),x) = 0, \tag{3.28}$$

$$\ell_1(\ell_3(x,y,\mathbf{f})) + \ell_2(\ell_2(x,y),\mathbf{f}) - \ell_2(\ell_2(x,\mathbf{f}),y) + \ell_2(\ell_2(y,\mathbf{f}),x) + \ell_3(\ell_1(\mathbf{f}),x,y) = 0, \tag{3.29}$$

$$\ell_2(\ell_2(x,y),\mathbf{a}) - \ell_2(\ell_2(x,\mathbf{a}),y) + \ell_2(\ell_2(y,\mathbf{a}),x) + \ell_3(\ell_1(\mathbf{a}),x,y) = 0, \tag{3.30}$$

$$\ell_2(\ell_2(x,\mathbf{f}),\mathbf{g}) + \ell_2(\ell_2(x,\mathbf{g}),\mathbf{f}) + \ell_2(\ell_2(\mathbf{f},\mathbf{g}),x) - \ell_3(\ell_1(\mathbf{f}),x,\mathbf{g}) - \ell_3(\ell_1(\mathbf{g}),x,\mathbf{f}) = 0. \tag{3.31}$$

It is easy to associate to any such map ℓ_3 a unique Jacobiator $\mathfrak{G}(\ell_3) = J$: it suffices to set $J_{xyz} := ([[x,y],z], \ell_3(x,y,z)) \in L_1$, for any $x,y,z \in L_0$. Equation (3.28) means that J_{xyz} has the correct target. Equations (3.25) and (3.26) exactly correspond to Equations (3.29) and (3.30), respectively, *if we assume that in total degree $d = 1$, ℓ_3 is valued in 2-cocycles and vanishes when evaluated on a 1-coboundary.* These conditions are verified if we start from a structure map ℓ_3 that vanishes on any arguments of total degree 1.

Remark 3.4.11. Remark that the values $\ell_3(x,y,\mathbf{f}) \in V_2$ cannot be encoded in a natural 2-transformation $J : L_0^{\times 3} \ni (x,y,z) \to J_{xyz} \in L_1$ (and that the same holds true for Equation (3.31), whose first three terms are zero, since we started from a map ℓ_2 that vanishes on $V_1 \times V_1$).

Proof of Lemma 3.4.10 (continuation). Conversely, to any Jacobiator J corresponds a unique map $\mathfrak{N}(J) = \ell_3$. Just set $\ell_3(x,y,z) := \mathbf{J}_{xyz} \in V_1$ and $\ell_3(x,y,\mathbf{f}) = 0$, for all $x,y,z \in V_0$ and $\mathbf{f} \in V_1$ (as ℓ_3 is required to have weight 1, it must vanish if evaluated on elements of degree $d \geq 2$).

Obviously the composites $\mathfrak{N}\mathfrak{G}$ and $\mathfrak{G}\mathfrak{N}$ are identity maps. □

Remark 3.4.12. The naturality condition is, roughly speaking, the categorical analogue of the L_∞-condition $n = 3$.

3.4.4 Identiator – fourth structure map

For $x,y,z,u \in L_0$, we set

$$\eta_{xyzu} := [J_{x,y,z}, 1_u] \circ_0 (J_{[x,z],y,u} + J_{x,[y,z],u}) \circ_0 ([J_{xzu}, 1_y] + 1) \circ_0 ([1_x, J_{yzu}] + 1) \in L_1 \tag{3.32}$$

and

$$\varepsilon_{xyzu} := J_{[x,y],z,u} \circ_0 ([J_{xyu}, 1_z] + 1) \circ_0 (J_{x,[y,u],z} + J_{[x,u],y,z} + J_{x,y,[z,u]}) \in L_1, \tag{3.33}$$

see Definition 3.3.3. The identities 1 are uniquely determined by the sources of the involved factors. The quadrilinear natural 2-transformations η and ε are actually the left and right hand composites of the Baez-Crans octagon that pictures the coherence law of a Lie 2-algebra, see [BC04], Definition 4.1.3. They connect the quadrilinear 2-functors $F, G : L \times L \times L \times L \to L$, whose values at (x,y,z,u) are given by the source and the target of the 1-cells η_{xyzu} and ε_{xyzu}, as well as by the top and bottom sums of triple brackets of the mentioned octagon.

Lemma 3.4.13. *The skew-symmetric quadrilinear 2-modifications $\mu : \eta \Rrightarrow \varepsilon$ are in 1-to-1 correspondence with the graded antisymmetric quadrilinear weight 2 maps $\ell_4 : V^{\times 4} \to V$ that verify the L_∞-condition $n = 4$.*

Proof. A skew-symmetric quadrilinear 2-modification $\mu : \eta \Rrightarrow \varepsilon$ maps every tuple $(x, y, z, u) \in L_0^{\times 4}$ to a unique $\mu_{xyzu} : \eta_{xyzu} \Rightarrow \varepsilon_{xyzu}$ in L_2, such that, for any $\alpha = (u, \mathbf{f}, \mathbf{a}) \in L_2$, we have

$$F(1_x^2, 1_y^2, 1_z^2, \alpha) \circ_0 \mu_{x,y,z,u+t\mathbf{f}} = \mu_{xyzu} \circ_0 G(1_x^2, 1_y^2, 1_z^2, \alpha) \tag{3.34}$$

(as well as similar results concerning naturality with respect to the three other variables). If we decompose $\mu_{xyzu} \in L_2 = V_0 \oplus V_1 \oplus V_2$,

$$\mu_{xyzu} = (F(x, y, z, u), \mathbf{h}_{xyzu}, \mathbf{m}_{xyzu}) = 1_{\eta_{xyzu}} + \mathbf{m}_{xyzu},$$

Condition (3.34) reads

$$F(1_x, 1_y, 1_z, \mathbf{f}) + \mathbf{h}_{x,y,z,u+t\mathbf{f}} = \mathbf{h}_{xyzu} + G(1_x, 1_y, 1_z, \mathbf{f}), \tag{3.35}$$

$$F(1_x^2, 1_y^2, 1_z^2, \mathbf{a}) + \mathbf{m}_{x,y,z,u+t\mathbf{f}} = \mathbf{m}_{xyzu} + G(1_x^2, 1_y^2, 1_z^2, \mathbf{a}). \tag{3.36}$$

On the other hand, a graded antisymmetric quadrilinear weight 2 map $\ell_4 : V^{\times 4} \to V$, and more precisely $\ell_4 : V_0^{\times 4} \to V_2$, verifies L_∞-condition $n = 4$, if

$$\begin{gathered}
\ell_1(\ell_4(a,b,c,d)) \\
-\ell_2(\ell_3(a,b,c),d) + (-1)^{cd}\ell_2(\ell_3(a,b,d),c) - (-1)^{b(c+d)}\ell_2(\ell_3(a,c,d),b) \\
+(-1)^{a(b+c+d)}\ell_2(\ell_3(b,c,d),a) + \ell_3(\ell_2(a,b),c,d) - (-1)^{bc}\ell_3(\ell_2(a,c),b,d) \\
+(-1)^{d(b+c)}\ell_3(\ell_2(a,d),b,c) + (-1)^{a(b+c)}\ell_3(\ell_2(b,c),a,d) \\
-(-1)^{ab+ad+cd}\ell_3(\ell_2(b,d),a,c) + (-1)^{(a+b)(c+d)}\ell_3(\ell_2(c,d),a,b) \\
-\ell_4(\ell_1(a),b,c,d) + (-1)^{ab}\ell_4(\ell_1(b),a,c,d) \\
-(-1)^{c(a+b)}\ell_4(\ell_1(c),a,b,d) + (-1)^{d(a+b+c)}\ell_4(\ell_1(d),a,b,c) = 0,
\end{gathered} \tag{3.37}$$

for all homogeneous $a, b, c, d \in V$. The condition is trivial for $d \geq 2$. For $d = 0$, we write $(a, b, c, d) = (x, y, z, u) \in V_0^{\times 4}$, and, for $d = 1$, we take $(a, b, c, d) = (x, y, z, \mathbf{f}) \in V_0^{\times 3} \times V_1$, so that – since ℓ_2 and ℓ_3 vanish on $V_1 \times V_1$ and for $d = 1$, respectively – Condition (3.37) reads

$$\ell_1(\ell_4(x, y, z, u)) - \mathbf{h}_{xyzu} + \mathbf{e}_{xyzu} = 0, \tag{3.38}$$

$$\ell_4(\ell_1(\mathbf{f}), x, y, z) = 0, \tag{3.39}$$

where $\mathbf{h}_{xyzu}$ and $\mathbf{e}_{xyzu}$ are the V_1-components of η_{xyzu} and ε_{xyzu}, see Equations (3.32) and (3.33).

We can associate to any such map ℓ_4 a unique 2-modification $\mathfrak{G}(\ell_4) = \mu$, $\mu : \eta \Rrightarrow \varepsilon$. It suffices to set, for $x, y, z, u \in L_0$,

$$\mu_{xyzu} = (F(x, y, z, u), \mathbf{h}_{xyzu}, -\ell_4(x, y, z, u)) \in L_2.$$

In view of Equation (3.38), the target of this 2-cell is

$$t\mu_{xyzu} = (F(x, y, z, u), \mathbf{h}_{xyzu} - \ell_1(\ell_4(x, y, z, u))) = \varepsilon_{xyzu} \in L_1.$$

Note now that the 2-naturality equations (3.25) and (3.26) show that 2-naturality of $\eta : F \Rightarrow G$ means that

$$F(1_x, 1_y, 1_z, \mathbf{f}) + \mathbf{h}_{x,y,z,u+t\mathbf{f}} = \mathbf{h}_{xyzu} + G(1_x, 1_y, 1_z, \mathbf{f}),$$
$$F(1_x^2, 1_y^2, 1_z^2, \mathbf{a}) = G(1_x^2, 1_y^2, 1_z^2, \mathbf{a}).$$

When comparing with Equations (3.35) and (3.36), we conclude that μ is a 2-modification if and only if $\ell_4(\ell_1(\mathbf{f}), x, y, z) = 0$, which is exactly Equation (3.39).

Conversely, if we are given a skew-symmetric quadrilinear 2-modification $\mu : \eta \Rrightarrow \varepsilon$, we define a map $\mathfrak{N}(\mu) = \ell_4$ by setting $\ell_4(x, y, z, u) = -\mathbf{m}_{xyzu}$, with self-explaining notations. L_∞-condition $n = 4$ is equivalent with Equations (3.38) and (3.39). The first means that μ_{xyzu} must have the target ε_{xyzu} and the second requires that $\mathbf{m}_{t\mathbf{f},x,y,z}$ vanish – a consequence of the 2-naturality of η and of Equation (3.36).

The maps $\mathfrak{N}$ and $\mathfrak{G}$ are again inverses. □

3.4.5 Coherence law – L_∞-condition $n = 5$

Lemma 3.4.14. *Coherence law (3.14) is equivalent to L_∞-condition $n = 5$.*

Proof. The sh Lie condition $n = 5$ reads,

$$\ell_2(\ell_4(x,y,z,u),v)-\ell_2(\ell_4(x,y,z,v),u)+\ell_2(\ell_4(x,y,u,v),z)-\ell_2(\ell_4(x,z,u,v),y)+\ell_2(\ell_4(y,z,u,v),x)$$
$$+\ell_4(\ell_2(x,y),z,u,v)-\ell_4(\ell_2(x,z),y,u,v)+\ell_4(\ell_2(x,u),y,z,v)-\ell_4(\ell_2(x,v)y,z,u)+\ell_4(\ell_2(y,z),x,u,v)$$
$$-\ell_4(\ell_2(y,u),x,z,v)+\ell_4(\ell_2(y,v),x,z,u)+\ell_4(\ell_2(z,u),x,y,v)-\ell_4(\ell_2(z,v),x,y,u)+\ell_4(\ell_2(u,v),x,y,z)$$
$$= 0, \tag{3.40}$$

for any $x, y, z, u, v \in V_0$. It is trivial in degree $d \geq 1$. Let us mention that it follows from Equation (3.28) that (V_0, ℓ_2) is a Lie algebra up to homotopy, and from Equation (3.30) that ℓ_2 is a representation of V_0 on V_2. Condition (3.40) then requires that ℓ_4 be a Lie algebra 4-cocycle of V_0 represented upon V_2.

The coherence law for the 2-modification μ corresponds to four different ways to rebracket the expression $F([x, y], z, u, v) = [[[[x, y], z], u], v]$ by means of μ, J, and $[-,-]$. More precisely, we define, for any tuple $(x, y, z, u, v) \in L_0^{\times 5}$, four 2-cells

$$\alpha_i : \sigma_i \Rightarrow \tau_i,$$

$i \in \{1, 2, 3, 4\}$, in L_2, where $\sigma_i, \tau_i : A_i \to B_i$. Dependence on the considered tuple is understood. We omit temporarily also index i. Of course, σ and τ read $\sigma = (A, \mathbf{s}) \in L_1$ and $\tau = (A, \mathbf{t}) \in L_1$.

$$\text{If } \alpha = (A, \mathbf{s}, \mathbf{a}) \in L_2, \text{ we set } \alpha^{-1} = (A, \mathbf{t}, -\mathbf{a}) \in L_2,$$

which is, as easily seen, the inverse of α for composition along 1-cells.

Definition 3.4.15. The **coherence law** for the 2-modification μ of a Lie 3-algebra $(L, [-,-], J, \mu)$ reads

$$\alpha_1 + \alpha_4^{-1} = \alpha_3 + \alpha_2^{-1}, \tag{3.41}$$

where $\alpha_1 - \alpha_4$ are detailed in the next definitions.

Definition 3.4.16. The **first 2-cell** α_1 is given by

$$\alpha_1 = 1_{11} \circ_0 \left(\mu_{x,y,z,[u,v]} + [\mu_{xyzv}, 1_u^2]\right) \circ_0 1_{12} \circ_0 \left(\mu_{[x,v],y,z,u} + \mu_{x,[y,v],z,u} + \mu_{x,y,[z,v],u} + 1^2\right), \quad (3.42)$$

where

$$1_{11} = 1_{J_{[[x,y],z],u,v}}, 1_{12} = 1_{[J_{x,[z,v],y},1_u]+[J_{[x,v],z,y},1_u]+[J_{x,z,[y,v]},1_u]} + 1^2, \quad (3.43)$$

and where the 1^2 are the identity 2-cells associated with the elements of L_0 provided by the composability condition.

For instance, the squared target of the second factor of α_1 is $G(x,y,z,[u,v]) + [G(x,y,z,v),u]$, whereas the squared source of the third factor is

$$[[[x,[z,v]],y],u] + [[[[x,v],z],y],u] + [[[x,z],[y,v]],u] + \ldots .$$

As the three first terms of this sum are three of the six terms of $[G(x,y,z,v),u]$, the object "...", at which 1^2 in 1_{12} is evaluated, is the sum of the remaining terms and $G(x,y,z,[u,v])$.

Definition 3.4.17. The **fourth 2-cell** α_4 is equal to

$$\alpha_4 = [\mu_{xyzu}, 1_v^2] \circ_0 1_{41} \circ_0 \left(\mu_{[x,u],y,z,v} + \mu_{x,[y,u],z,v} + \mu_{x,y,[z,u],v}\right) \circ_0 1_{42}, \quad (3.44)$$

where

$$1_{41} = 1_{[J_{[x,u],z,y},1_v]+[J_{x,z,[y,u]},1_v]+[J_{x,[z,u],y},1_v]} + 1^2,$$

$$1_{42} = 1_{[[J_{xuv},1_z],1_y]+[J_{xuv},1_{[y,z]}]+[1_x,[J_{yuv},1_z]]+[[1_x,J_{zuv}],1_y]+[1_x,[1_y,J_{zuv}]]+[1_{[x,z]},J_{yuv}]} + 1^2. \quad (3.45)$$

Definition 3.4.18. The **third 2-cell** α_3 reads

$$\alpha_3 = \mu_{[x,y],z,u,v} \circ_0 1_{31} \circ_0 \left([\mu_{xyuv}, 1_z^2] + 1^2\right) \circ_0 1_{32} \circ_0 1_{33}, \quad (3.46)$$

where

$$1_{31} = 1_{[J_{[x,y],v,u},1_z]} + 1^2,$$

$$1_{32} = 1_{[J_{xyv},1_{[z,u]}]+J_{x,y,[[z,v],u]}+J_{x,y,[z,[u,v]]}+J_{[[x,v],u],y,z}+J_{[x,v],[y,u],z}+J_{[x,u],[y,v],z}+J_{x,[[y,v],u],z}+J_{[x,[u,v]],y,z}+J_{x,[y,[u,v]],z}+[J_{xyu},1_{[z,v]}]},$$

$$1_{33} = 1_{J_{x,[y,v],[z,u]}+J_{[x,v],y,[z,u]}+J_{x,[y,u],[z,v]}+J_{[x,u],y,[z,v]}} + 1^2. \quad (3.47)$$

Definition 3.4.19. The **second 2-cell** α_2 is defined as

$$\alpha_2 = 1_{21} \circ_0 \left(\mu_{[x,z],y,u,v} + \mu_{x,[y,z],u,v}\right) \circ_0 1_{22} \circ_0 \left([1_x^2, \mu_{yzuv}] + [\mu_{xzuv}, 1_y^2] + 1^2\right) \circ_0 1_{23}, \quad (3.48)$$

where

$$1_{21} = 1_{[[J_{xyz},1_u],1_v]}, 1_{22} = 1_{[1_x,J_{[y,z],v,u}]+[J_{[x,z],v,u},1_y]} + 1^2,$$

$$1_{23} = 1_{[J_{xzv},1_{[y,u]}]+[J_{xzu},1_{[y,v]}]+[1_{[x,v]},J_{yzu}]+[1_{[x,u]},J_{yzv}]} + 1^2. \quad (3.49)$$

To get the component expression

$$(A_1 + A_4, \mathbf{s}_1 + \mathbf{t}_4, \mathbf{a}_1 - \mathbf{a}_4) = (A_3 + A_2, \mathbf{s}_3 + \mathbf{t}_2, \mathbf{a}_3 - \mathbf{a}_2) \quad (3.50)$$

of the coherence law (3.41), we now comment on the computation of the components $(A_i, \mathbf{s}_i, \mathbf{a}_i)$ (resp. $(A_i, \mathbf{t}_i, -\mathbf{a}_i)$) of α_i (resp. α_i^{-1}).

As concerns α_1, it is straightforwardly seen that all compositions make sense, that its V_0-component is

$$A_1 = F([x,y],z,u,v),$$

and that the V_2-component is

$$\mathbf{a}_1 =$$

$$-\ell_4(x,y,z,\ell_2(u,v)) - \ell_2(\ell_4(x,y,z,v),u) - \ell_4(\ell_2(x,v),y,z,u) - \ell_4(x,\ell_2(y,v),z,u) - \ell_4(x,y,\ell_2(z,v),u).$$

When actually examining the composability conditions, we find that 1^2 in the fourth factor of α_1 is $1^2_{G(x,y,z,[u,v])}$ and thus that the target $t^2\alpha_1$ is made up by the 24 terms

$$G([x,v],y,z,u) + G(x,[y,v],z,u) + G(x,y,[z,v],u) + G(x,y,z,[u,v]).$$

The computation of the V_1-component $\mathbf{s}_1$ is tedious but simple – it leads to a sum of 29 terms of the type "$\ell_3\ell_2\ell_2$, $\ell_2\ell_3\ell_2$, or $\ell_2\ell_2\ell_3$". We will comment on it in the case of α_4^{-1}, which is slightly more interesting.

The V_0-component of α_4^{-1} is

$$A_4 = [F_{xyzu}, v] = F([x,y],z,u,v)$$

and its V_2-component is equal to

$$-\mathbf{a}_4 = \ell_2(\ell_4(x,y,z,u),v) + \ell_4(\ell_2(x,u),y,z,v) + \ell_4(x,\ell_2(y,u),z,v) + \ell_4(x,y,\ell_2(z,u),v).$$

The V_1-component $\mathbf{t}_4$ of α_4^{-1} is the V_1-component of the target of α_4. This target is the composition of the targets of the four factors of α_4 and its V_1-component is given by

$$\mathbf{t}_4 = [\mathbf{e}_{xyzu}, 1_v] + [\mathbf{J}_{[x,u],z,y}, 1_v] + [\mathbf{J}_{x,z,[y,u]}, 1_v] + [\mathbf{J}_{x,[z,u],y}, 1_v] + \mathbf{e}_{[x,u],y,z,v} + \mathbf{e}_{x,[y,u],z,v} + \mathbf{e}_{x,y,[z,u],v}$$

$$+[[\mathbf{J}_{xuv}, 1_z], 1_y] + [\mathbf{J}_{xuv}, 1_{[y,z]}] + [1_x, [\mathbf{J}_{yuv}, 1_z]] + [[1_x, \mathbf{J}_{zuv}], 1_y] + [1_x, [1_y, \mathbf{J}_{zuv}]] + [1_{[x,z]}, \mathbf{J}_{yuv}].$$

The definition (3.33) of ε immediately provides its V_1-component $\mathbf{e}$ as a sum of 5 terms of the type "$\ell_3\ell_2$ or $\ell_2\ell_3$". The preceding V_1-component $\mathbf{t}_4$ of α_4^{-1} can thus be explicitly written as a sum of 29 terms of the type "$\ell_3\ell_2\ell_2$, $\ell_2\ell_3\ell_2$, or $\ell_2\ell_2\ell_3$". It can moreover be checked that the target $t^2\alpha_4^{-1}$ is again a sum of 24 terms – the same as for $t^2\alpha_1$.

The V_0-component of α_3 is

$$A_3 = F([x,y],z,u,v),$$

the V_1-component $\mathbf{s}_3$ can be computed as before and is a sum of 25 terms of the usual type "$\ell_3\ell_2\ell_2$, $\ell_2\ell_3\ell_2$, or $\ell_2\ell_2\ell_3$", whereas the V_2-component is equal to

$$\mathbf{a}_3 = -\ell_4(\ell_2(x,y),z,u,v) - \ell_2(\ell_4(x,y,u,v),z).$$

Again $t^2\alpha_3$ is made up by the same 24 terms as $t^2\alpha_1$ and $t^2\alpha_4^{-1}$.

Eventually, the V_0-component of α_2^{-1} is

$$A_2 = F([x,y],z,u,v),$$

the V_1-component $\mathbf{t}_2$ is straightforwardly obtained as a sum of 27 terms of the form "$\ell_3\ell_2\ell_2$, $\ell_2\ell_3\ell_2$, or $\ell_2\ell_2\ell_3$", and the V_2-component reads

$$-\mathbf{a}_2 = \ell_4(\ell_2(x,z),y,u,v) + \ell_4(x,\ell_2(y,z),u,v) + \ell_2(x,\ell_4(y,z,u,v)) + \ell_2(\ell_4(x,z,u,v),y).$$

The target $t^2\alpha_2^{-1}$ is the same as in the preceding cases.

Coherence condition (3.41) and its component expression (3.50) can now be understood. The condition on the V_0-components is obviously trivial. The condition on the V_2-components is nothing but L_∞-condition $n = 5$, see Equation (3.40). The verification of triviality of the condition on the V_1-components is lengthy: 6 pairs (resp. 3 pairs) of terms of the LHS $\mathbf{s}_1 + \mathbf{t}_4$ (resp. RHS $\mathbf{s}_3 + \mathbf{t}_2$) are opposite and cancel out, 25 terms of the LHS coincide with terms of the RHS, and, finally, 7 triplets of LHS-terms combine with triplets of RHS-terms and provide 7 sums of 6 terms, e.g.

$$\ell_3(\ell_2(\ell_2(x,y),z),u,v) + \ell_2(\ell_3(x,y,z),\ell_2(u,v)) + \ell_2(\ell_2(\ell_3(x,y,z),v),u)$$

$$-\ell_2(\ell_2(\ell_3(x,y,z),u),v) - \ell_3(\ell_2(\ell_2(x,z),y),u,v) - \ell_3(\ell_2(x,\ell_2(y,z)),u,v).$$

Since, for $\mathbf{f} = \ell_3(x,y,z) \in V_1$, we have

$$\ell_1(\mathbf{f}) = t\mathbf{J}_{xyz} = \ell_2(\ell_2(x,z),y) + \ell_2(x,\ell_2(y,z)) - \ell_2(\ell_2(x,y),z),$$

the preceding sum vanishes in view of Equation (3.29). Indeed, if we associate a Lie 3-algebra to a 3-term Lie infinity algebra, we started from a homotopy algebra whose term ℓ_3 vanishes in total degree 1, and if we build an sh algebra from a categorified algebra, we already constructed an ℓ_3-map with that property. Finally, the condition on V_1-components is really trivial and the coherence law (3.41) is actually equivalent to L_∞-condition $n = 5$. □

3.5 Monoidal structure of the category $\texttt{Vect}\ n\text{-}\texttt{Cat}$

In this section we exhibit a specific aspect of the natural monoidal structure of the category of linear n-categories.

Proposition 3.5.1. *If L and L' are linear n-categories, a family $F_m : L_m \to L'_m$ of linear maps that respects sources, targets, and identities, commutes automatically with compositions and thus defines a linear n-functor $F : L \to L'$.*

Proof. If $v = (v_0, \ldots, v_m), w = (w_0, \ldots, w_m) \in L_m$ are composable along a p-cell, then $F_m v = (F_0 v_0, \ldots, F_m v_m)$ and $F_m w = (F_0 w_0, \ldots, F_m w_m)$ are composable as well, and $F_m(v \circ_p w) = (F_m v) \circ_p (F_m w)$ in view of Equation (3.6). □

Proposition 3.5.2. *The category* $\texttt{Vect}\ n\text{-}\texttt{Cat}$ *admits a canonical symmetric monoidal structure* $\boxtimes$.

Proof. We first define the product $\boxtimes$ of two linear n-categories L and L'. The n-globular vector space that underlies the linear n-category $L \boxtimes L'$ is defined in the obvious way, $(L \boxtimes L')_m = L_m \otimes L'_m$, $S_m = s_m \otimes s'_m$, $T_m = t_m \otimes t'_m$. Identities are clear as well, $I_m = 1_m \otimes 1'_m$. These data can be completed by the unique possible compositions $\square_p$ that then provide a linear n-categorical structure.

If $F : L \to M$ and $F' : L' \to M'$ are two linear n-functors, we set

$$(F \boxtimes F')_m = F_m \otimes F'_m \in \mathrm{Hom}_{\mathbb{K}}(L_m \otimes L'_m, M_m \otimes M'_m),$$

where $\mathbb{K}$ denotes the ground field. Due to Proposition 3.5.1, the family $(F \boxtimes F')_m$ defines a linear n-functor $F \boxtimes F' : L \boxtimes L' \to M \boxtimes M'$.
It is immediately checked that $\boxtimes$ respects composition and is therefore a functor from the product category $(\texttt{Vect}\ n\text{-}\texttt{Cat})^{\times 2}$ to $\texttt{Vect}\ n\text{-}\texttt{Cat}$. Further, the linear n-category K, defined by $K_m = \mathbb{K}$, $s_m = t_m = \mathrm{id}_{\mathbb{K}}$ $(m > 0)$, and $1_m = \mathrm{id}_{\mathbb{K}}$ $(m < n)$, acts as identity object for $\boxtimes$. Its is now clear that $\boxtimes$ endows $\texttt{Vect}\ n\text{-}\texttt{Cat}$ with a symmetric monoidal structure. □

Proposition 3.5.3. *Let L, L', and L'' be linear n-categories. For any bilinear n-functor $F : L \times L' \to L''$, there exists a unique linear n-functor $\tilde{F} : L \boxtimes L' \to L''$, such that $\boxtimes \tilde{F} = F$. Here $\boxtimes : L \times L' \to L \boxtimes L'$ denotes the family of bilinear maps $\boxtimes_m : L_m \times L'_m \ni (v, v') \mapsto v \otimes v' \in L_m \otimes L'_m$, and juxtaposition denotes the obvious composition of the first with the second factor.*

Proof. The result is a straightforward consequence of the universal property of the tensor product of vector spaces. □

The next remark is essential.

Remark 3.5.4. Proposition 3.5.3 is not a Universal Property for the tensor product $\boxtimes$ of $\texttt{Vect}$ n-$\texttt{Cat}$, since $\boxtimes : L \times L' \to L \boxtimes L'$ is not a bilinear n-functor. It follows that bilinear n-functors on a product category $L \times L'$ cannot be identified with linear n-functors on the corresponding tensor product category $L \boxtimes L'$.

The point is that the family $\boxtimes_m$ of bilinear maps respects sources, targets, and identities, but not compositions (in contrast with a similar family of linear maps, see Proposition 3.5.1). Indeed, if $(v, v'), (w, w') \in L_m \times L'_m$ are two p-composable pairs (note that this condition is equivalent with the requirement that $v, w \in L_m$ and $v', w' \in L'_m$ be p-composable), we have

$$\boxtimes_m((v, v') \circ_p (w, w')) = (v \circ_p w) \otimes (v' \circ_p w') \in L_m \otimes L'_m, \tag{3.51}$$

and

$$\boxtimes_m(v, v') \ \circ_p\ \boxtimes_m(w, w') = (v \otimes v') \circ_p (w \otimes w') \in L_m \otimes L'_m. \tag{3.52}$$

As the elements (3.51) and (3.52) arise from the compositions in $L_m \times L'_m$ and $L_m \otimes L'_m$, respectively, – which are forced by linearity and thus involve the completely different linear structures of these spaces – it can be expected that the two elements do not coincide.

Indeed, when confining ourselves, to simplify, to the case $n = 1$ of linear categories, we easily check that

$$(v \circ w) \otimes (v' \circ w') = (v \otimes v') \circ (w \otimes w') + (v - 1_{tv}) \otimes \mathbf{w}' + \mathbf{w} \otimes (v' - 1_{tv'}). \tag{3.53}$$

Observe also that the source spaces of the linear maps

$$\circ_L \otimes \circ'_{L'} : (L_1 \times_{L_0} L_1) \otimes (L'_1 \times_{L'_0} L'_1) \ni (v, w) \otimes (v', w') \mapsto (v \circ w) \otimes (v' \circ w') \in L_1 \otimes L'_1$$

and

$$\circ_{L \boxtimes L'} : (L_1 \otimes L'_1) \times_{L_0 \otimes L'_0} (L_1 \otimes L'_1) \ni ((v \otimes v'), (w \otimes w')) \mapsto (v \otimes v') \circ (w \otimes w') \in L_1 \otimes L'_1$$

are connected by

$$\ell_2 : (L_1 \times_{L_0} L_1) \otimes (L'_1 \times_{L'_0} L'_1) \ni (v, w) \otimes (v', w') \mapsto (v \otimes v', w \otimes w') \in (L_1 \otimes L'_1) \times_{L_0 \otimes L'_0} (L_1 \otimes L'_1) \tag{3.54}$$

– a linear map with nontrivial kernel.

3.6 Discussion

We continue working in the case $n = 1$ and investigate a more conceptual approach to the construction of a chain map $\ell_2 : \mathfrak{N}(L) \otimes \mathfrak{N}(L) \to \mathfrak{N}(L)$ from a bilinear functor $[-,-] : L \times L \to L$.

When denoting by $[-,-] : L \boxtimes L \to L$ the induced linear functor, we get a chain map $\mathfrak{N}([-,-]) : \mathfrak{N}(L \boxtimes L) \to \mathfrak{N}(L)$, so that it is natural to look for a second chain map

$$\phi : \mathfrak{N}(L) \otimes \mathfrak{N}(L) \to \mathfrak{N}(L \boxtimes L).$$

The informed reader may skip the following subsection.

3.6.1 Nerve and normalization functors, Eilenberg-Zilber chain map

The objects of the simplicial category Δ are the finite ordinals $n = \{0, \ldots, n-1\}$, $n \geq 0$. Its morphisms $f : m \to n$ are the order respecting functions between the sets m and n. Let $\delta_i : n \rightarrowtail n+1$ be the injection that omits image i, $i \in \{0, \ldots, n\}$, and let $\sigma_i : n+1 \twoheadrightarrow n$ be the surjection that assigns the same image to i and $i+1$, $i \in \{0, \ldots, n-1\}$. Any order respecting function $f : m \to n$ reads uniquely as $f = \sigma_{j_1} \ldots \sigma_{j_h} \delta_{i_1} \ldots \delta_{i_k}$, where the j_r are decreasing and the i_s increasing. The application of this epi-monic decomposition to binary composites $\delta_i \delta_j$, $\sigma_i \sigma_j$, and $\delta_i \sigma_j$ yields three basic commutation relations.

A simplicial object in the category $\texttt{Vect}$ is a functor $S \in [\Delta^{+\mathrm{op}}, \texttt{Vect}]$, where Δ^+ denotes the full subcategory of Δ made up by the nonzero finite ordinals. We write this functor $n+1 \mapsto S(n+1) =: S_n$, $n \geq 0$, (S_n is the vector space of n-simplices), $\delta_i \mapsto S(\delta_i) =: d_i : S_n \to S_{n-1}$, $i \in \{0, \ldots, n\}$ (d_i is a face operator), $\sigma_i \mapsto S(\sigma_i) =: s_i : S_n \to S_{n+1}$, $i \in \{0, \ldots, n\}$ (s_i is a degeneracy operator). The d_i and s_j verify the duals of the mentioned commutation rules. The simplicial data (S_n, d_i^n, s_i^n) (we added superscript n) of course completely determine the functor S. Simplicial objects in $\texttt{Vect}$ form themselves a category, namely the functor category $s(\texttt{Vect}) := [\Delta^{+\mathrm{op}}, \texttt{Vect}]$, for which the morphisms, called simplicial morphisms, are the natural transformations between such functors. In view of the epi-monic factorization, a simplicial map $\alpha : S \to T$ is exactly a family of linear maps $\alpha_n : S_n \to T_n$ that commute with the face and degeneracy operators.

The nerve functor

$$\mathcal{N} : \texttt{VectCat} \to s(\texttt{Vect})$$

is defined on a linear category L as the sequence $L_0, L_1, L_2 := L_1 \times_{L_0} L_1, L_3 := L_1 \times_{L_0} L_1 \times_{L_0} L_1 \ldots$ of vector spaces of $0, 1, 2, 3 \ldots$ simplices, together with the face operators "composition" and the degeneracy operators "insertion of identity", which verify the simplicial commutation rules. Moreover, any linear functor $F : L \to L'$ defines linear maps $F_n : L_n \ni (v_1, \ldots, v_n) \to (F(v_1), \ldots, F(v_n)) \in L'_n$ that implement a simplicial map.

The normalized or Moore chain complex of a simplicial vector space $S = (S_n, d_i^n, s_i^n)$ is given by $N(S)_n = \cap_{i=1}^n \ker d_i^n \subset S_n$ and $\partial_n = d_0^n$. Normalization actually provides a functor

$$N : s(\mathtt{Vect}) \leftrightarrow \mathtt{C}^+(\mathtt{Vect}) : \Gamma$$

valued in the category of nonnegatively graded chain complexes of vector spaces. Indeed, if $\alpha : S \to T$ is a simplicial map, then $\alpha_{n-1} d_i^n = d_i^n \alpha_n$. Thus, $N(\alpha) : N(S) \to N(T)$, defined on $c_n \in N(S)_n$ by $N(\alpha)_n(c_n) = \alpha_n(c_n)$, is valued in $N(T)_n$ and is further a chain map. Moreover, the Dold-Kan correspondence claims that the normalization functor N admits a right adjoint Γ and that these functors combine into an equivalence of categories.

It is straightforwardly seen that, for any linear category L, we have

$$N(\mathcal{N}(L)) = \mathfrak{N}(L). \tag{3.55}$$

The categories $s(\mathtt{Vect})$ and $\mathtt{C}^+(\mathtt{Vect})$ have well-known monoidal structures (we denote the unit objects by I_s and $I_{\mathtt{C}}$, respectively). The normalization functor $N : s(\mathtt{Vect}) \to \mathtt{C}^+(\mathtt{Vect})$ is lax monoidal, i.e. it respects the tensor products and unit objects up to coherent chain maps $\varepsilon : I_{\mathtt{C}} \to N(I_s)$ and

$$EZ_{S,T} : N(S) \otimes N(T) \to N(S \otimes T)$$

(functorial in $S, T \in s(\mathtt{Vect})$), where $EZ_{S,T}$ is the Eilenberg-Zilber map. Functor N is lax comonoidal or oplax monoidal as well, the chain morphism being here the Alexander-Whitney map $AW_{S,T}$. These chain maps are inverses of each other up to chain homotopy, $EZ\ AW = 1,\ AW\, EZ \sim 1$.

The Eilenberg-Zilber map is defined as follows. Let $a \otimes b \in N(S)_p \otimes N(T)_q \subset S_p \otimes T_q$ be an element of degree $p+q$. The chain map $EZ_{S,T}$ sends $a \otimes b$ to an element of $N(S \otimes T)_{p+q} \subset (S \otimes T)_{p+q} = S_{p+q} \otimes T_{p+q}$. We have

$$EZ_{S,T}(a \otimes b) = \sum_{(p,q)-\text{shuffles } (\mu,\nu)} \operatorname{sign}(\mu,\nu)\ \ s_{\nu_q}(\ldots(s_{\nu_1} a)) \otimes s_{\mu_p}(\ldots(s_{\mu_1} b)) \in S_{p+q} \otimes T_{p+q},$$

where the shuffles are permutations of $(0, \ldots, p+q-1)$ and where the s_i are the degeneracy operators.

3.6.2 Monoidal structure and obstruction

We now come back to the construction of a chain map $\phi : \mathfrak{N}(L) \otimes \mathfrak{N}(L) \to \mathfrak{N}(L \boxtimes L)$.

For $L' = L$, the linear map (3.54) reads

$$\ell_2 : (\mathcal{N}(L) \otimes \mathcal{N}(L))_2 \ni (v, w) \otimes (v', w') \mapsto (v \otimes v', w \otimes w') \in \mathcal{N}(L \boxtimes L)_2.$$

If its obvious extensions ℓ_n to all other spaces $(\mathcal{N}(L) \otimes \mathcal{N}(L))_n$ define a simplicial map $\ell : \mathcal{N}(L) \otimes \mathcal{N}(L) \to \mathcal{N}(L \boxtimes L)$, then

$$N(\ell) : N(\mathcal{N}(L) \otimes \mathcal{N}(L)) \to N(\mathcal{N}(L \boxtimes L))$$

is a chain map. Its composition with the Eilenberg-Zilber chain map

$$EZ_{\mathcal{N}(L),\mathcal{N}(L)} : N(\mathcal{N}(L)) \otimes N(\mathcal{N}(L)) \to N(\mathcal{N}(L) \otimes \mathcal{N}(L))$$

finally provides the searched chain map ϕ, see Equation (3.55).
However, the ℓ_n do not commute with all degeneracy and face operators. Indeed, we have for instance

$$\ell_2((d_2^3 \otimes d_2^3)((u,v,w) \otimes (u',v',w'))) = (u \otimes u', (v \circ w) \otimes (v' \circ w')),$$

whereas

$$d_2^3(\ell_3((u,v,w) \otimes (u',v',w'))) = (u \otimes u', (v \otimes v') \circ (w \otimes w')).$$

Equation (3.53), which means that $\boxtimes : L \times L' \to L \boxtimes L'$ is not a functor, shows that these results do not coincide.

A natural idea would be to change the involved monoidal structures $\boxtimes$ of $\mathtt{VectCat}$ or $\otimes$ of $\mathtt{C}^+(\mathtt{Vect})$. However, even if we substitute the Loday-Pirashvili tensor product $\otimes_{\mathrm{LP}}$ of 2-term chain complexes of vector spaces, i.e. of linear maps [LP98], for the usual tensor product $\otimes$, we do not get $\mathfrak{N}(L) \otimes_{\mathrm{LP}} \mathfrak{N}(L) = \mathfrak{N}(L \boxtimes L)$.

4 The Supergeometry of Loday Algebroids

The next research paper was published in 'Journal of Geometric Mechanics', 5(2) (2013), 185-213 (joint work with Janusz Grabowski and Norbert Poncin).

4.1 Introduction

The concept of *Dirac structure*, proposed by Dorfman [Dor87] in the Hamiltonian framework of integrable evolution equations and defined in [Cou90] as an isotropic subbundle of the Whitney sum $\mathcal{T}M = TM \oplus_M T^*M$ of the tangent and the cotangent bundles and satisfying some additional conditions, provides a geometric setting for Dirac's theory of constrained mechanical systems. To formulate the integrability condition defining the Dirac structure, Courant [Cou90] introduced a natural skew-symmetric bracket operation on sections of $\mathcal{T}M$. The Courant bracket does not satisfy the Leibniz rule with respect to multiplication by functions nor the Jacobi identity. These defects disappear upon restriction to a Dirac subbundle because of the isotropy condition. Particular cases of Dirac structures are graphs of closed 2-forms and Poisson bivector fields on the manifold M.

The nature of the Courant bracket itself remained unclear until several years later when it was observed by Liu, Weinstein and Xu [LWX97] that $\mathcal{T}M$ endowed with the Courant bracket plays the role of a 'double' object, in the sense of Drinfeld [Dr86], for a pair of Lie algebroids (see [Mck05]) over M. Let us recall that, in complete analogy with Drinfeld's Lie bialgebras, in the category of Lie algebroids there also exist 'bi-objects', Lie bialgebroids, introduced by Mackenzie and Xu [MX94] as linearizations of Poisson groupoids. On the other hand, every Lie bialgebra has a double which is a Lie algebra. This is not so for general Lie bialgebroids. Instead, Liu, Weinstein and Xu [LWX97] showed that the double of a Lie bialgebroid is a more complicated structure they call a *Courant algebroid*, $\mathcal{T}M$ with the Courant bracket being a special case.

There is also another way of viewing Courant algebroids as a generalization of Lie algebroids. This requires a change in the definition of the Courant bracket and considering an analog of the non-antisymmetric Dorfman bracket [Dor87], so that the traditional Courant bracket becomes the skew-symmetrization of the new one [Roy02]. This change replaces one of the defects with another one: a version of the Jacobi identity is satisfied, while the bracket is no longer skew-symmetric. Such algebraic structures have been introduced by Loday [Lod93] under the name *Leibniz algebras*, but they are nowadays also often called *Loday algebras*. Loday algebras, like their skew-symmetric counterparts – Lie algebras – determine certain cohomological complexes, defined on tensor algebras instead of Grassmann algebras. Canonical examples of Loday algebras arise often as *derived brackets* introduced by Kosmann-Schwarzbach [KSc96, KSc04].

Since Loday brackets, like the Courant-Dorfman bracket, appear naturally in Geometry and Physics in the form of 'algebroid brackets', i.e. brackets on sections of vector bundles, there were several attempts to formalize the concept of *Loday* (or *Leibniz*) *algebroid* (see e.g. [Bar12, BV11, Gra03, GM01, ILMP99, Hag02, HM02, KS10, MM05, SX08, Wa02]). We prefer the terminology *Loday algebroid* to distinguish them from other *general algebroid* brackets with both anchors (see [GU99]), called sometimes *Leibniz algebroids* or *Leibniz brackets* and used recently in Physics, for instance, in the context of nonholonomic constraints [GG08, GG11, GGU06, GLMM09, OPB04]. Note also that a Loday algebroid is the horizontal categorification of a Loday algebra; vertical categorification would lead to Loday n-algebras, which are tightly related to truncated Loday infinity algebras, see [AP10], [KMP11].

The concepts of Loday algebroid we found in the literature do not seem to be exactly appropriate. The notion in [Gra03], which assumes the existence of both anchor maps, is too strong and admits no real new examples, except for Lie algebroids and bundles of Loday algebras. The concept introduced in [SX08] requires a pseudo-Riemannian metric on the bundle, so it is too strong as well and does not reduce to a Loday algebra when we consider a bundle over a single point, while the other concepts [Hag02, HM02, ILMP99, KS10, MM05, Wa02], assuming only the existence of a left anchor, do not put any differentiability requirements for the first variable, so that they are not geometric and too weak (see Example 4.4.3). Only in [Bar12] one considers some Leibniz algebroids with local brackets.
The aim of this work is to propose a modified concept of Loday algebroid in terms of an operation on sections of a vector bundle, as well as in terms of a homological vector field of a supercommutative manifold. We put some minimal requirements that a proper concept of Loday algebroid should satisfy. Namely, the definition of Loday algebroid, understood as a certain operation on sections of a vector bundle E,

- should reduce to the definition of Loday algebra in the case when E is just a vector space;
- should contain the Courant-Dorfman bracket as a particular example;
- should be as close to the definition of Lie algebroid as possible.

We propose a definition satisfying all these requirements and including all main known examples of Loday brackets with geometric origins. Moreover, we can interpret our Loday algebroid structures as homological vector fields on a supercommutative manifold; this opens, like in the case of Lie algebroids, new horizons for a geometric understanding of these objects and of their possible 'higher generalizations' [BP12]. This supercommutative manifold is associated with a superalgebra of differential operators, whose multiplication is a supercommutative shuffle product.
Note that we cannot work with the supermanifold ΠE like in the case of a Lie algebroid on E, since the Loday coboundary operator rises the degree of a differential operator, even for Lie algebroid brackets. For instance, the Loday differential associated with the standard bracket of vector fields produces the Levi-Civita connection out of a Riemannian metric [Ldd04]. However, the Levi-Civita connection $\nabla_X Z$ is no longer a tensor, as it is of the first-order with respect to Z. Therefore, instead of the Grassmann algebra $\mathrm{Sec}(\wedge E^*)$ of 'differential forms', which are zero-degree skew-symmetric multidifferential operators on E, we are forced to consider, not just the tensor algebra of sections of $\oplus_{k=0}^{\infty}(E^*)^{\otimes k}$, but the algebra $\mathcal{D}^\bullet(E)$ spanned by all multidifferential operators

$$D : \mathrm{Sec}(E) \times \cdots \times \mathrm{Sec}(E) \to C^\infty(M)\,.$$

However, to retain the supergeometric flavor, we can reduce ourselves to a smaller subspace $\mathrm{D}^\bullet(E)$ of $\mathcal{D}^\bullet(E)$, which is a subalgebra with respect to the canonical supercommutative shuffle product and is closed under the Loday coboundary operators associated with the Loday algebroids we introduce. This interesting observation deserves further investigations that we postpone to a next paper.
We should also make clear that, although the algebraic structures in question have their roots in Physics (see the papers on Geometric Mechanics mentioned above), we do not propose in this paper new applications to Physics, but focus on finding a proper framework unifying all these

structures. Our work seems to be technically complicated enough and applications to Mechanics will be the subject of a separate work.

The paper is organized as follows. We first recall, in Section 4.2, needed results on differential operators and derivative endomorphisms. In Section 4.3 we investigate, under the name of pseudoalgebras, algebraic counterparts of algebroids requiring varying differentiability properties for the two entries of the bracket. The results of Section 4.4 show that we should relax our traditional understanding of the right anchor map. A concept of Loday algebroid satisfying all the above requirements is proposed in Definition 4.4.7 and further detailed in Theorem 4.4.8. In Section 4.5 we describe a number of new Loday algebroids containing main canonical examples of Loday brackets on sections of a vector bundle. A natural reduction a Loday pseudoalgebra to a Lie pseudoalgebra is studied in Section 4.6. For the standard Courant bracket it corresponds to its reduction to the Lie bracket of vector fields. We then define Loday algebroid cohomology, Section 4.7, and interpret in Section 4.8 our Loday algebroid structures in terms of homological vector fields of the graded ringed space given by the shuffle multiplication of multidifferential operators, see Theorem 4.8.6. We introduce also the corresponding Cartan calculus.

4.2 Differential operators and derivative endomorphisms

All geometric objects, like manifolds, bundles, maps, sections, etc. will be smooth throughout this paper.

Definition 4.2.1. A *Lie algebroid* structure on a vector bundle $\tau : E \to M$ is a Lie algebra bracket $[\cdot,\cdot]$ on the real vector space $\mathcal{E} = \mathrm{Sec}(E)$ of sections of E which satisfies the following compatibility condition related to the $\mathcal{A} = C^\infty(M)$-module structure in $\mathcal{E}$:

$$\forall\ X, Y \in \mathcal{E}\ \forall f \in \mathcal{A} \quad [X, fY] - f[X, Y] = \rho(X)(f)Y\,, \tag{4.1}$$

for some vector bundle morphism $\rho : E \to TM$ covering the identity on M and called the *anchor map*. Here, $\rho(X) = \rho \circ X$ is the vector field on M associated *via* ρ with the section X.

Note that the bundle morphism ρ is uniquely determined by the bracket of the Lie algebroid. What differs a general Lie algebroid bracket from just a Lie module bracket on the $C^\infty(M)$-module $\mathrm{Sec}(E)$ of sections of E is the fact that it is not $\mathcal{A}$-bilinear but a certain first-order bidifferential operator: the adjoint operator $\mathrm{ad}_X = [X,\cdot]$ is a *derivative endomorphism*, i.e., the *Leibniz rule*

$$\mathrm{ad}_X(fY) = f\mathrm{ad}_X(Y) + \widehat{X}(f)Y \tag{4.2}$$

is satisfied for each $Y \in \mathcal{E}$ and $f \in \mathcal{A}$, where $\widehat{X} = \rho(X)$ is the vector field on M assigned to X, the *anchor* of X. Moreover, the assignment $X \mapsto \widehat{X}$ is a differential operator of order 0, as it comes from a bundle map $\rho : E \mapsto TM$.
Derivative endomorphisms (also called *quasi-derivations*), like differential operators in general, can be defined for any module $\mathcal{E}$ over an associative commutative ring $\mathcal{A}$. Also an extension to superalgebras is straightforward. These natural ideas go back to Grothendieck and Vinogradov [Vin72]. On the module $\mathcal{E}$ we have namely a distinguished family $\mathcal{A}_\mathcal{E} = \{f_\mathcal{E} : f \in \mathcal{A}\}$ of linear operators provided by the module structure: $f_\mathcal{E}(Y) = fY$.

Definition 4.2.2. Let $\mathcal{E}_i$, $i = 1, 2$, be modules over the same ring $\mathcal{A}$. We say that an additive operator $D : \mathcal{E}_1 \to \mathcal{E}_2$ is a *differential operator of order 0*, if it intertwines $f_{\mathcal{E}_1}$ with $f_{\mathcal{E}_2}$, i.e.

$$\delta(f)(D) := D \circ f_{\mathcal{E}_1} - f_{\mathcal{E}_2} \circ D\,, \tag{4.3}$$

vanishes for all $f \in \mathcal{A}$. Inductively, we say that D is a *differential operator of order* $\leq k+1$, if the commutators (4.3) are differential operators of order $\leq k$. In other words, D is a differential operator of order $\leq k$ if and only if

$$\forall\ f_1, \ldots, f_{k+1} \in \mathcal{A} \quad \delta(f_1)\delta(f_2)\cdots\delta(f_{k+1})(D) = 0\,. \tag{4.4}$$

The corresponding set of differential operators of order $\leq k$ will be denoted by $\mathcal{D}_k(\mathcal{E}_1; \mathcal{E}_2)$ (shortly, $\mathcal{D}_k(\mathcal{E})$, if $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$) and the set of differential operators of arbitrary order (filtered by $(\mathcal{D}_k(\mathcal{E}_1; \mathcal{E}_2))_{k=0}^{\infty}$) by $\mathcal{D}(\mathcal{E}_1; \mathcal{E}_2)$ (resp., $\mathcal{D}(\mathcal{E})$). We will say that D *is of order* k if it is of order $\leq k$ and not of order $\leq k-1$.

In particular, $\mathcal{D}_0(\mathcal{E}_1; \mathcal{E}_2) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}_1; \mathcal{E}_2)$ is made up by module homomorphisms. Note that in the case when $\mathcal{E}_i = \mathrm{Sec}(E_i)$ is the module of sections of a vector bundle E_i, $i = 1, 2$, the concept of differential operators defined above coincides with the standard understanding. As this will be our standard geometric model, to reduce algebraic complexity we will assume that $\mathcal{A}$ is an associative commutative algebra with unity 1 over a field $\mathbb{K}$ of characteristic 0 and all the $\mathcal{A}$-modules are faithful. In this case, $\mathcal{D}(\mathcal{E}_1; \mathcal{E}_2)$ is a (canonically filtered) vector space over $\mathbb{K}$ and, since we work with fields of characteristic 0, condition (4.4) is equivalent to a simpler condition (see [Gra92])

$$\forall\ f \in \mathcal{A} \quad \delta(f)^{k+1}(D) = 0\,. \tag{4.5}$$

If $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, then $\delta(f)(D) = [D, f_{\mathcal{E}}]_c$, where $[\cdot,\cdot]_c$ is the commutator bracket, and elements of $\mathcal{A}_{\mathcal{E}}$ are particular 0-order operators. Therefore, we can canonically identify $\mathcal{A}$ with the subspace $\mathcal{A}_{\mathcal{E}}$ in $\mathcal{D}_0(\mathcal{E})$ and use it to distinguish a particular set of first-order differential operators on $\mathcal{E}$ as follows.

Definition 4.2.3. *Derivative endomorphisms* (or *quasi-derivations*) $D : \mathcal{E} \to \mathcal{E}$ are particular first-order differential operators distinguished by the condition

$$\forall\ f \in \mathcal{A} \quad \exists\ \widehat{f} \in \mathcal{A} \quad [D, f_{\mathcal{E}}]_c = \widehat{f}_{\mathcal{E}}\,. \tag{4.6}$$

Since the commutator bracket satisfies the Jacobi identity, one can immediately conclude that $\widehat{f}_{\mathcal{E}} = \widehat{D}(f)_{\mathcal{E}}$ which holds for some derivation $\widehat{D} \in \mathrm{Der}(\mathcal{A})$ and an arbitrary $f \in \mathcal{A}$ [Gra03]. Derivative endomorphisms form a submodule $\mathrm{Der}(\mathcal{E})$ in the $\mathcal{A}$-module $\mathrm{End}_{\mathbb{K}}(\mathcal{E})$ of $\mathbb{K}$-linear endomorphisms of $\mathcal{E}$ which is simultaneously a Lie subalgebra over $\mathbb{K}$ with respect to the commutator bracket. The linear map,

$$\mathrm{Der}(\mathcal{E}) \ni D \mapsto \widehat{D} \in \mathrm{Der}(\mathcal{A})\,,$$

called the *universal anchor map*, is a differential operator of order 0, $\widehat{fD} = f\widehat{D}$. The Jacobi identity for the commutator bracket easily implies (see [Gra03, Theorem 2])

$$\widehat{[D_1, D_2]_c} = [\widehat{D}_1, \widehat{D}_1]_c\,. \tag{4.7}$$

It is worth remarking (see [Gra03]) that also $\mathcal{D}(\mathcal{E})$ is a Lie subalgebra in $\mathrm{End}_{\mathbb{K}}(\mathcal{E})$, as

$$[\mathcal{D}_k(\mathcal{E}), \mathcal{D}_l(\mathcal{E})]_c \subset \mathcal{D}_{k+l-1}(\mathcal{E})\,, \tag{4.8}$$

and an associative subalgebra, as

$$\mathcal{D}_k(\mathcal{E}) \circ \mathcal{D}_l(\mathcal{E}) \subset \mathcal{D}_{k+l}(\mathcal{E})\,, \tag{4.9}$$

that makes $\mathcal{D}(\mathcal{E})$ into a canonical example of a *quantum Poisson algebra* in the terminology of [GP04].

It was pointed out in [KSM02] that the concept of derivative endomorphism can be traced back to N. Jacobson [Jac35, Jac37] as a special case of his *pseudo-linear endomorphism*. It has appeared also in [Nel67] under the name *module derivation* and was used to define linear connections in the algebraic setting. In the geometric setting of Lie algebroids it has been studied in [Mck05] under the name *covariant differential operator*. For more detailed history and recent development we refer to [KSM02].

Algebraic operations in differential geometry have usually a local character in order to be treatable with geometric methods. On the pure algebraic level we should work with differential (or multidifferential) operations, as tells us the celebrated Peetre Theorem [Pe59, Pe60]. The algebraic concept of a multidifferential operator is obvious. For a $\mathbb{K}$-multilinear operator $D:\mathcal{E}_1\times\cdots\times\mathcal{E}_p\to\mathcal{E}$ and each $i=1,\dots,p$, we say that D is a *differential operator of order* $\le k$ *with respect to the* i*th variable*, if, for all $y_j\in\mathcal{E}_j$, $j\ne i$,

$$D(y_1,\dots,y_{i-1},\,\cdot\,,y_{i+1},\dots,y_p):\mathcal{E}_i\to\mathcal{E}$$

is a differential operator of order $\le k$. In other words,

$$\forall\ f\in\mathcal{A}\quad \delta_i(f)^{k+1}(D)=0\,, \tag{4.10}$$

where

$$\delta_i(f)D(y_1,\dots,y_p)=D(y_1,\dots,fy_i,\dots,y_p)-fD(y_1,\dots,y_p)\,. \tag{4.11}$$

Note that the operations $\delta_i(f)$ and $\delta_j(g)$ commute. We say that the operator D *is a multidifferential operator of order* $\le n$, if it is of order $\le n$ with respect to each variable separately. This means that, fixing any $p-1$ arguments, we get a differential operator of order $\le n$. A similar, but stronger, definition is the following

Definition 4.2.4. We say that a multilinear operator $D:\mathcal{E}_1\times\cdots\times\mathcal{E}_p\to\mathcal{E}$ is a *multidifferential operator of total order* $\le k$, if

$$\forall\ f_1,\dots,f_{k+1}\in\mathcal{A}\ \forall\ i_1,\dots,i_{k+1}=1,\dots,p\quad \left[\delta_{i_1}(f_1)\delta_{i_2}(f_2)\cdots\delta_{i_{k+1}}(f_{k+1})(D)=0\right]\,. \tag{4.12}$$

Of course, a multidifferential operator of total order $\le k$ is a multidifferential operator of order $\le k$. It is also easy to see that a p-linear differential operator of order $\le k$ is a multidifferential operator of total order $\le pk$. In particular, the Lie bracket of vector fields (in fact, any Lie algebroid bracket) is a bilinear differential operator of total order ≤ 1.

4.3 Pseudoalgebras

Let us start this section with recalling that Loday, while studying relations between Hochschild and cyclic homology in the search for obstructions to the periodicity of algebraic K-theory, discovered that one can skip the skew-symmetry assumption in the definition of Lie algebra, still having a possibility to define an appropriate (co)homology (see [Lod92, LP93] and [Lod93, Chapter 10.6]). His Jacobi identity for such structures was formally the same as the classical Jacobi identity in the form

$$[x,[y,z]]=[[x,y],z]+[y,[x,z]]. \tag{4.13}$$

This time, however, this is no longer equivalent to

$$[[x, y], z] = [[x, z], y] + [x, [y, z]], \tag{4.14}$$

nor to

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \tag{4.15}$$

since we have no skew-symmetry. Loday called such structures *Leibniz algebras*, but to avoid collision with another concept of *Leibniz brackets* in the literature, we shall call them *Loday algebras*. This is in accordance with the terminology of [KSc96], where analogous structures in the graded case are defined. Note that the identities (4.13) and (4.14) have an advantage over the identity (4.15) obtained by cyclic permutations, since they describe the algebraic facts that the left-regular (resp., right-regular) actions are left (resp., right) derivations. This was the reason to name the structure 'Leibniz algebra'.

Of course, there is no particular reason not to define Loday algebras by means of (4.14) instead of (4.13) (and in fact, it was the original definition by Loday), but this is not a substantial difference, as both categories are equivalent via transposition of arguments. We will use the form (4.13) of the Jacobi identity.

Our aim is to find a proper generalization of the concept of Loday algebra in a way similar to that in which Lie algebroids generalize Lie algebras. If one thinks about a generalization of a concept of Lie algebroid as operations on sections of a vector bundle including operations (brackets) which are non-antisymmetric or which do not satisfy the Jacobi identity, and are not just $\mathcal{A}$-bilinear, then it is reasonable, on one hand, to assume differentiability properties of the bracket as close to the corresponding properties of Lie algebroids as possible and, on the other hand, including all known natural examples of such brackets. This is not an easy task, since, as we will see soon, some natural possibilities provide only few new examples.

To present a list of these possibilities, we propose the following definitions serving in the pure algebraic setting.

Definition 4.3.1. Let $\mathcal{E}$ be a faithful module over an associative commutative algebra $\mathcal{A}$ over a field $\mathbb{K}$ of characteristic 0. A a $\mathbb{K}$-bilinear bracket $B = [\cdot, \cdot] : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ on the module $\mathcal{E}$

1. is called a *faint pseudoalgebra bracket*, if B is a bidifferential operator;
2. is called a *weak pseudoalgebra bracket*, if B is a bidifferential operator of degree ≤ 1;
3. is called a *quasi pseudoalgebra bracket*, if B is a bidifferential operator of total degree ≤ 1;
4. is called a *pseudoalgebra bracket*, if B is a bidifferential operator of total degree ≤ 1 and the *adjoint map* $\mathrm{ad}_X = [X, \cdot] : \mathcal{E} \to \mathcal{E}$ is a derivative endomorphism for each $X \in \mathcal{E}$;
5. is called a *QD-pseudoalgebra bracket*, if the *adjoint maps* $\mathrm{ad}_X, \mathrm{ad}^r_X : \mathcal{E} \to \mathcal{E}$,

$$\mathrm{ad}_X = [X, \cdot]\,, \quad \mathrm{ad}^r_X = [\cdot, X] \quad (X \in \mathcal{E})\,, \tag{4.16}$$

associated with B are derivative endomorphisms (quasi-derivations);

6. is called a *strong pseudoalgebra bracket*, if B is a bidifferential operator of total degree ≤ 1 and the *adjoint maps* $\mathrm{ad}_X, \mathrm{ad}^r_X : \mathcal{E} \to \mathcal{E}$,

$$\mathrm{ad}_X = [X, \cdot]\,, \quad \mathrm{ad}^r_X = [\cdot, X] \quad (X \in \mathcal{E})\,, \tag{4.17}$$

are derivative endomorphisms.

We call the module $\mathcal{E}$ equipped with such a bracket, respectively, a *faint pseudoalgebra*, *weak pseudoalgebra* etc. If the bracket is symmetric (skew-symmetric), we speak about faint, weak, etc., *symmetric* (*skew*) *pseudoalgebras*. If the bracket satisfies the Jacobi identity (4.13), we speak about local, weak, etc., *Loday pseudoalgebras*, and if the bracket is a Lie algebra bracket, we speak about local, weak, etc., *Lie pseudoalgebras*. If $\mathcal{E}$ is the $\mathcal{A} = C^\infty(M)$ module of sections of a vector bundle $\tau : E \to M$, we refer to the above pseudoalgebra structures as to *algebroids*.

Theorem 4.3.2. *If* $[\cdot,\cdot]$ *is a pseudoalgebra bracket, then the map*

$$\rho : \mathcal{E} \to \mathrm{Der}(\mathcal{A}), \quad \rho(X) = \widehat{\mathrm{ad}_X},$$

called the anchor map, *is* $\mathcal{A}$*-linear,* $\rho(fX) = f\rho(X)$*, and*

$$[X, fY] = f[X,Y] + \rho(X)(f)Y \tag{4.18}$$

for all $X, Y \in \mathcal{E}$*,* $f \in \mathcal{A}$*. Moreover, if* $[\cdot,\cdot]$ *satisfies additionally the Jacobi identity, i.e., we deal with a Loday pseudoalgebra, then the anchor map is a homomorphism into the commutator bracket,*

$$\rho\left([X,Y]\right) = [\rho(X), \rho(Y)]_c\,. \tag{4.19}$$

Proof. Since the bracket B is a bidifferential operator of total degree ≤ 1, we have $\delta_1(f)\delta_2(g)B = 0$ for all $f, g \in \mathcal{A}$. On the other hand, as easily seen,

$$(\delta_1(f)\delta_2(g)B)(X,Y) = \left(\rho(fX) - f\rho(X)\right)(g)Y\,, \tag{4.20}$$

and the module is faithful, it follows $\rho(fX) = f\rho(X)$. The identity (4.19) is a direct implication of the Jacobi identity combined with (4.18).

□

Theorem 4.3.3. *If* $[\cdot,\cdot]$ *is a QD-pseudoalgebra bracket, then it is a weak pseudoalgebra bracket and admits two anchor maps*

$$\rho, \rho^r : \mathcal{E} \to \mathrm{Der}(\mathcal{A}), \quad \rho(X) = \widehat{\mathrm{ad}_X}\,, \ \rho^r = -\widehat{\mathrm{ad}^r}\,,$$

for which we have

$$[X, fY] = f[X,Y] + \rho(X)(f)Y\,, \quad [fX, Y] = f[X,Y] - \rho^r(X)(f)Y\,, \tag{4.21}$$

for all $X, Y \in \mathcal{E}$*,* $f \in \mathcal{A}$*. If the bracket is skew-symmetric, then both anchors coincide, and if the bracket is a strong QD-pseudoalgebra bracket, they are* $\mathcal{A}$*-linear. Moreover, if* $[\cdot,\cdot]$ *satisfies additionally the Jacobi identity, i.e., we deal with a Loday QD-pseudoalgebra, then, for all* $X, Y \in \mathcal{E}$*,*

$$\rho\left([X,Y]\right) = [\rho(X), \rho(Y)]_c\,. \tag{4.22}$$

Proof. Similarly as above,

$$(\delta_2(f)\delta_2(g)B)(X,Y) = \rho(X)(g)fY - f\rho(X)(g)Y = 0\,,$$

so B is a first-order differential operator with respect to the second argument. The same can be done for the first argument.

Next, as for any QD-pseudoalgebra bracket B we have, analogously to (4.20),

$$(\delta_1(f)\delta_2(g)B)(X,Y) = (\rho(fX) - f\rho(X))\,(g)Y = (\rho^r(gY) - g\rho^r(Y))\,(f)X\,, \tag{4.23}$$

both anchor maps are $\mathcal{A}$-linear if and only if D is of total order ≤ 1. The rest follows analogously to the previous theorem.

□

The next observation is that quasi pseudoalgebra structures on an $\mathcal{A}$-module $\mathcal{E}$ have certain analogs of anchor maps, namely $\mathcal{A}$-module homomorphisms $b = b^r, b^l : \mathcal{E} \to \mathrm{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \mathrm{End}(\mathcal{E})$. For every $X \in \mathcal{E}$ we will view $b(X)$ as an $\mathcal{A}$-module homomorphism $b(X) : \Omega^1 \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$, where Ω^1 is the $\mathcal{A}$-submodule of $\mathrm{Hom}_{\mathcal{A}}(\mathrm{Der}(\mathcal{A}); \mathcal{A})$ generated by $\mathrm{d}\mathcal{A} = \{\mathrm{d}f : f \in \mathcal{A}\}$ and $\langle \mathrm{d}f, D\rangle = D(f)$. Elements of $\mathrm{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \mathrm{End}(\mathcal{E})$ act on elements of $\Omega^1 \otimes_{\mathcal{A}} \mathcal{E}$ in the obvious way: $(V \otimes \Phi)(\omega \otimes X) = \langle V, \omega\rangle \Phi(X)$.

Theorem 4.3.4. *A $\mathbb{K}$-bilinear bracket $B = [\cdot,\cdot]$ on an $\mathcal{A}$-module $\mathcal{E}$ defines a quasi pseudoalgebra structure if and only if there are $\mathcal{A}$-module homomorphisms*

$$b^r, b^l : \mathcal{E} \to \mathrm{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \mathrm{End}(\mathcal{E})\,, \tag{4.24}$$

called generalized anchor maps, right and left*, such that, for all $X, Y \in \mathcal{E}$ and all $f \in \mathcal{A}$,*

$$[X, fY] = f[X,Y] + b^l(X)(\mathrm{d}f \otimes Y)\,, \quad [fX, Y] = f[X,Y] - b^r(Y)(\mathrm{d}f \otimes X)\,. \tag{4.25}$$

The generalized anchor maps are actual anchor maps if they take values in $\mathrm{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \{\mathrm{Id}_{\mathcal{E}}\}$.

Proof. Assume first that the bracket B is a bidifferential operator of total degree ≤ 1 and define a three-linear map of vector spaces $A : \mathcal{E} \times \mathcal{A} \times \mathcal{E} \to \mathcal{E}$ by

$$A(X, g, Y) = (\delta_2(g)B)(X,Y) = [X, gY] - g[X,Y]\,.$$

It is easy to see that A is $\mathcal{A}$-linear with respect to the first and the third argument, and a derivation with respect to the second. Indeed, as

$$(\delta_1(f)\delta_2(g)B)(X,Y) = A(fX, g, Y) - fA(X, g, Y) = 0\,,$$

we get $\mathcal{A}$-linearity with respect to the first argument. Similarly, from $\delta_2(f)\delta_2(g)B = 0$, we get the same conclusion for the third argument. We have also

$$\begin{aligned} A(X, fg, Y) &= [X, fgY] - fg[X,Y] = [X, fgY] - f[X, gY] + f[X, gY] - fg[X,Y] \\ &= A(X, f, gY) + fA(X, g, Y) = gA(X, f, Y) + fA(X, g, Y)\,, \end{aligned} \tag{4.26}$$

thus the derivation property. This implies that A is represented by an $\mathcal{A}$-module homomorphism $b^l : \mathcal{E} \to \mathrm{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \mathrm{End}(\mathcal{E})$. Analogous considerations give us the right generalized anchor map b^r.
Conversely, assume the existence of both generalized anchor maps. Then, the map A defined as above reads $A(X, f, Y) = b^l(X)(\mathrm{d}f \otimes Y)$, so is $\mathcal{A}$-linear with respect to X and Y. Hence,

$$(\delta_1(f)\delta_2(g)B)(X,Y) = A(fX, g, Y) - fA(X, g, Y) = 0$$

and

$$(\delta_2(f)\delta_2(g)B)(X,Y) = A(X, g, fY) - fA(X, g, Y) = 0\,.$$

A similar reasoning for b^r gives $(\delta_1(f)\delta_1(g)B)(X,Y) = 0$, so the bracket is a bidifferential operator of total order ≤ 1. □

In the case when we deal with a quasi algebroid, i.e., $\mathcal{A} = C^\infty(M)$ and $\mathcal{E} = \mathrm{Sec}(E)$ for a vector bundle $\tau : E \to M$, the generalized anchor maps (4.24) are associated with vector bundle maps that we denote (with some abuse of notations) also by b^r, b^l,

$$b^r, b^l : E \to TM \otimes_M \mathrm{End}(E)\,,$$

covering the identity on M. Here, $\mathrm{End}(E)$ is the endomorphism bundle of E, so $\mathrm{End}(E) \simeq E^* \otimes_M E$. The induced maps of sections produce from sections of E sections of $TM \otimes_M \mathrm{End}(E)$ which, in turn, act on sections of $T^*M \otimes_M E$ in the obvious way. An algebroid version of Theorem 4.3.4 is the following.

Theorem 4.3.5. *An $\mathbb{R}$-bilinear bracket $B = [\cdot,\cdot]$ on the real space* $\mathrm{Sec}(E)$ *of sections of a vector bundle $\tau : E \to M$ defines a quasi algebroid structure if and only if there are vector bundle morphisms*

$$b^r, b^l : E \to TM \otimes_M \mathrm{End}(E) \tag{4.27}$$

covering the identity on M, called generalized anchor maps, right and left*, such that, for all $X, Y \in \mathrm{Sec}(E)$ and all $f \in C^\infty(M)$, (4.25) is satisfied. The generalized anchor maps are actual anchor maps, if they take values in $TM \otimes \langle \mathrm{Id}_E \rangle \simeq TM$.*

4.4 Loday algebroids

Let us isolate and specify the most important particular cases of Definition 4.3.1.

Definition 4.4.1.

1. A *faint Loday algebroid* (resp., *faint Lie algebroid*) on a vector bundle E over a base manifold M is a Loday bracket (resp., a Lie bracket) on the $C^\infty(M)$-module $\mathrm{Sec}(E)$ of smooth sections of E which is a bidifferential operator.

2. A *weak Loday algebroid* (resp., *weak Lie algebroid*) on a vector bundle E over a base manifold M is a Loday bracket (resp., a Lie bracket) on the $C^\infty(M)$-module $\mathrm{Sec}(E)$ of smooth sections of E which is a bidifferential operator of degree ≤ 1 with respect to each variable separately.

3. A *Loday quasi algebroid* (resp., *Lie quasi algebroid*) on a vector bundle E over a base manifold M is a Loday bracket (resp., Lie bracket) on the $C^\infty(M)$-module $\mathrm{Sec}(E)$ of smooth sections of E which is a bidifferential operator of total degree ≤ 1.

4. A *QD-algebroid* (resp., *skew QD-algebroid, Loday QD-algebroid, Lie QD-algebroid*) on a vector bundle E over a base manifold M is an $\mathbb{R}$-bilinear bracket (resp., skew bracket, Loday bracket, Lie bracket) on the $C^\infty(M)$-module $\mathrm{Sec}(E)$ of smooth sections of E for which the adjoint operators ad_X and ad^r_X are derivative endomorphisms.

Remark 4.4.2. *Lie pseudoalgebras* appeared first in the paper of Herz [Her53], but one can find similar concepts under more than a dozen of names in the literature (e.g. *Lie modules, (R, A)-Lie algebras, Lie-Cartan pairs, Lie-Rinehart algebras, differential algebras*, etc.). Lie algebroids were introduced by Pradines [Pra67] as infinitesimal parts of differentiable groupoids. In the same year a book by Nelson was published where a general theory of Lie modules, together with a big

part of the corresponding differential calculus, can be found. We also refer to a survey article by Mackenzie [Mck95]. QD-algebroids, as well as Loday QD-algebroids and Lie QD-algebroids, have been introduced in [Gra03]. In [GU99, GGU06] Loday strong QD-algebroids have been called Loday algebroids and strong QD-algebroids have been called just *algebroids*. The latter served as geometric framework for generalized Lagrange and Hamilton formalisms.
In the case of line bundles, $\mathrm{rk}E = 1$, Lie QD-algebroids are exactly *local Lie algebras* in the sense of Kirillov [Kir76]. They are just *Jacobi brackets*, if the bundle is trivial, $\mathrm{Sec}(E) = C^\infty(M)$. Of course, Lie QD-algebroid brackets are first-order bidifferential operators by definition, while Kirillov has originally started with considering Lie brackets on sections of line bundles determined by local operators and has only later discovered that these operators have to be bidifferential operators of first order. A purely algebraic version of Kirillov's result has been proven in [Gra92], Theorems 4.2 and 4.4, where bidifferential Lie brackets on associative commutative algebras containing no nilpotents have been considered.

Example 4.4.3. Let us consider a Loday algebroid bracket in the sense of [Hag02, HM02, ILMP99, KS10, MM05, Wa02], i.e., a Loday algebra bracket $[\cdot,\cdot]$ on the $C^\infty(M)$-module $\mathcal{E} = \mathrm{Sec}(E)$ of sections of a vector bundle $\tau : E \to M$ for which there is a vector bundle morphism $\rho : E \to TM$ covering the identity on M (the left anchor map) such that (4.1) is satisfied. Since, due to (4.19), the anchor map is necessarily a homomorphism of the Loday bracket into the Lie bracket of vector fields, our Loday algebroid is just a Lie algebroid in the case when ρ is injective. In the other cases the anchor map does not determine the Loday algebroid structure, in particular does not imply any locality of the bracket with respect to the first argument. Thus, this concept of Loday algebroid is not geometric.
For instance, let us consider a Whitney sum bundle $E = E_1 \oplus_M E_2$ with the canonical projections $p_i : E \to E_i$ and any $\mathbb{R}$-linear map $\varphi : \mathrm{Sec}(E_1) \to C^\infty(M)$. Being only $\mathbb{R}$-linear, φ can be chosen very strange non-geometric and non-local. Define now the following bracket on $\mathrm{Sec}(E)$:

$$[X, Y] = \varphi(p_1(X)) \cdot p_2(Y)\,.$$

It is easy to see that this is a Loday bracket which admits the trivial left anchor, but the bracket is non-local and non-geometric as well.

Example 4.4.4. A standard example of a weak Lie algebroid bracket is a Poisson (or, more generally, Jacobi) bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ viewed as a $C^\infty(M)$-module of section of the trivial line bundle $M \times \mathbb{R}$. It is a bidifferential operator of order ≤ 1 and the total order ≤ 2. It is actually a Lie QD-algebroid bracket, as ad_f and ad^r_f are, by definition, derivations (more generally, first-order differential operators). Both anchor maps coincide and give the corresponding Hamiltonian vector fields, $\rho(f)(g) = \{f, g\}$. The map $f \mapsto \rho(f)$ is again a differential operator of order 1, so is not implemented by a vector bundle morphism $\rho : M \times \mathbb{R} \to TM$. Therefore, this weak Lie algebroid is not a Lie algebroid. This has a straightforward generalization to *Kirillov brackets* being local Lie brackets on sections of a line bundle [Kir76].

Example 4.4.5. Various brackets are associated with a volume form ω on a manifold M of dimension n (see e.g. [Li74]). Denote with $\mathcal{X}^k(M)$ (resp., $\Omega^k(M)$) the spaces of k-vector fields (resp., k-forms) on M. As the contraction maps $\mathcal{X}^k(M) \ni K \mapsto i_K\omega \in \Omega^{n-k}(M)$ are isomorphisms of $C^\infty(M)$-modules, to the de Rham cohomology operator $\mathrm{d} : \Omega^{n-k-1}(M) \to \Omega^{n-k}(M)$ corresponds a homology operator $\delta : \mathcal{X}^k(M) \to \mathcal{X}^{k-1}(M)$. The skew-symmetric bracket B on $\mathcal{X}^2(M)$ defined in [Li74] by $B(t, u) = -\delta(t) \wedge \delta(u)$ is not a Lie bracket, since

its Jacobiator $B(B(t,u),v) + c.p.$ equals $\delta(\delta(t) \wedge \delta(u) \wedge \delta(v))$. A solution proposed in [Li74] depends on considering the algebra N of bivector fields modulo δ-exact bivector fields for which the Jacobi anomaly disappears, so that N is a Lie algebra.
Another option is to resign from skew-symmetry and define the corresponding faint Loday algebroid. In view of the duality between $\mathcal{X}^2(M)$ and Ω^{n-2}, it is possible to work with $\Omega^{n-2}(M)$ instead. For $\gamma \in \Omega^{n-2}(M)$ we define the vector field $\widehat{\gamma} \in \mathcal{X}(M)$ from the formula $i_{\widehat{\gamma}}\omega = \mathrm{d}\gamma$. The bracket in $\Omega^{n-2}(M)$ is now defined by (see [Lod93])

$$\{\gamma, \beta\}_\omega = \mathcal{L}_{\widehat{\gamma}}\beta = i_{\widehat{\gamma}} i_{\widehat{\beta}}\omega + \mathrm{d} i_{\widehat{\gamma}}\beta \,.$$

Since we have

$$i_{[\widehat{\gamma},\widehat{\beta}]_{vf}}\omega = \mathcal{L}_{\widehat{\gamma}} i_{\widehat{\beta}}\omega - i_{\widehat{\beta}}\mathcal{L}_{\widehat{\gamma}}\omega = \mathrm{d} i_{\widehat{\gamma}} i_{\widehat{\beta}}\omega = \mathrm{d}\{\gamma, \beta\}_\omega \,,$$

it holds

$$\widehat{\{\gamma, \beta\}_\omega} = [\widehat{\gamma}, \widehat{\beta}]_{vf} \,.$$

Therefore,

$$\{\{\gamma, \beta\}_\omega, \eta\}_\omega = \mathcal{L}_{\widehat{\{\gamma,\beta\}_\omega}}\eta = \mathcal{L}_{\widehat{\gamma}}\mathcal{L}_{\widehat{\beta}}\eta - \mathcal{L}_{\widehat{\beta}}\mathcal{L}_{\widehat{\gamma}}\eta = \{\gamma, \{\beta, \eta\}_\omega\}_\omega - \{\beta, \{\gamma, \eta\}_\omega\}_\omega \,,$$

so the Jacobi identity is satisfied and we deal with a Loday algebra. This is in fact a faint Loday algebroid structure on $\wedge^{n-2}T^*M$ with the left anchor $\rho(\gamma) = \widehat{\gamma}$. This bracket is a bidifferential operator which is first-order with respect to the second argument and second-order with respect to the first one.

Note that Lie QD-algebroids are automatically Lie algebroids, if the rank of the bundle E is > 1 [Gra03, Theorem 3]. Also some other of the above concepts do not produce qualitatively new examples.

Theorem 4.4.6. *([Gra03, GM01, GM03a])*

(a) Any Loday bracket on $C^\infty(M)$ (more generally, on sections of a line bundle) which is a bidifferential operator is actually a Jacobi bracket (first-order and skew-symmetric).

(b) Let $[\cdot, \cdot]$ be a Loday bracket on sections of a vector bundle $\tau : E \to M$, admitting anchor maps $\rho, \rho^r : \mathrm{Sec}(E) \to \mathcal{X}(M)$ which assign vector fields to sections of E and such that (4.21) is satisfied (Loday QD-algebroid on E). Then, the anchors coincide, $\rho = \rho^r$, and the bracket is skew-symmetric at points $p \in M$ in the support of $\rho = \rho^r$. Moreover, if the rank of E is > 1, then the anchor maps are $C^\infty(M)$-linear, i.e. they come from a vector bundle morphism $\rho = \rho^r : E \to M$. In other words, any Loday QD-algebroid is actually, around points where one anchor does not vanish, a Jacobi bracket if $\mathrm{rk}(E) = 1$, or Lie algebroid bracket if $\mathrm{rk}(E) > 1$.

The above results show that relaxing skew-symmetry and considering Loday brackets on $C^\infty(M)$ or $\mathrm{Sec}(E)$ does not lead to new structures (except for just bundles of Loday algebras), if we assume differentiability in the first case and the existence of both (possibly different) anchor maps in the second. Therefore, a definition of Loday algebroids that admits a rich family of new examples, must resign from the traditionally understood right anchor map.

The definition of the main object of our studies can be formulated as follows.

Definition 4.4.7. A *Loday algebroid* on a vector bundle E over a base manifold M is a Loday bracket on the $C^\infty(M)$-module $\mathrm{Sec}(E)$ of smooth sections of E which is a bidifferential operator of total degree ≤ 1 and for which the adjoint operator ad_X is a derivative endomorphism.

Of course, the above definition of Loday algebroid is stronger than those known in the literature (e.g. [Hag02, HM02, ILMP99, KS10, MM05, Wa02]), which assume only the existence of a left anchor and put no differentiability requirements for the first variable.

Theorem 4.4.8. *A Loday bracket $[\cdot,\cdot]$ on the real space $\mathrm{Sec}(E)$ of sections of a vector bundle $\tau : E \to M$ defines a Loday algebroid structure if and only if there are vector bundle morphisms*

$$\rho : E \to TM\,, \quad \alpha : E \to TM \otimes_M \mathrm{End}(E)\,, \tag{4.28}$$

covering the identity on M, such that, for all $X, Y \in \mathrm{Sec}(E)$ and all $f \in C^\infty(M)$,

$$[X, fY] = f[X,Y] + \rho(X)(f)Y\,, \quad [fX, Y] = f[X,Y] - \rho(Y)(f)X + \alpha(Y)(\mathrm{d}f \otimes X)\,. \tag{4.29}$$

If this is the case, the anchors are uniquely determined and the left anchor induces a homomorphism of the Loday bracket into the bracket $[\cdot,\cdot]_{vf}$ of vector fields,

$$\rho([X,Y]) = [\rho(X), \rho(Y)]_{vf}\,.$$

Proof. This is a direct consequence of Theorem 4.3.5 and the fact that an algebroid bracket has the left anchor map. We just write the generalized right anchor map as $b^r = \rho \otimes I - \alpha$. □

To give a local form of a Loday algebroid bracket, let us recall that sections X of the vector bundle E can be identified with linear (along fibers) functions ι_X on the dual bundle E^*. Thus, fixing local coordinates (x^a) in M and a basis of local sections e_i of E, we have a corresponding system $(x^a, \xi_i = \iota_{e_i})$ of affine coordinates in E^*. As local sections of E are identified with linear functions $\sigma = \sigma^i(x)\xi_i$, the Loday bracket is represented by a bidifferential operator B of total order ≤ 1:

$$B(\sigma_1^i(x)\xi_i, \sigma_2^j(x)\xi_j) = c_{ij}^k(x)\sigma_1^i(x)\sigma_2^j(x)\xi_k + \beta_{ij}^{ak}(x)\frac{\partial\sigma_1^i}{\partial x^a}(x)\sigma_2^j(x)\xi_k + \gamma_{ij}^{ak}(x)\sigma_1^i(x)\frac{\partial\sigma_2^j}{\partial x^a}(x)\xi_k\,.$$

Taking into account the existence of the left anchor, we have

$$\begin{aligned} B(\sigma_1^i(x)\xi_i, \sigma_2^j(x)\xi_j) \;=\;& c_{ij}^k(x)\sigma_1^i(x)\sigma_2^j(x)\xi_k + \alpha_{ij}^{ak}(x)\frac{\partial\sigma_1^i}{\partial x^a}(x)\sigma_2^j(x)\xi_k \\ &+\rho_i^a(x)\left(\sigma_1^i(x)\frac{\partial\sigma_2^j}{\partial x^a}(x) - \frac{\partial\sigma_1^j}{\partial x^a}(x)\sigma_2^i(x)\right)\xi_j\,. \end{aligned} \tag{4.30}$$

Since sections of $\mathrm{End}(E)$ can be written in the form of linear differential operators, we can rewrite (4.30) in the form

$$B = c_{ij}^k(x)\xi_k\partial_{\xi_i} \otimes \partial_{\xi_j} + \alpha_{ij}^{ak}(x)\xi_k\partial_{x^a}\partial_{\xi_i} \otimes \partial_{\xi_j} + \rho_i^a(x)\partial_{\xi_i} \wedge \partial_{x^a}\,. \tag{4.31}$$

Of course, there are additional relations between coefficients of B due to the fact that the Jacobi identity is satisfied.

4.5 Examples

4.5.1 Leibniz algebra

Of course, a finite-dimensional Leibniz algebra is a Leibniz algebroid over a point.

4.5.2 Courant-Dorfman bracket

The *Courant bracket* is defined on sections of $\mathcal{T}M = TM \oplus_M T^*M$ as follows:

$$[X+\omega, Y+\eta] = [X,Y]_{vf} + \mathcal{L}_X\eta - \mathcal{L}_Y\omega - \frac{1}{2}\left(d\, i_X\eta - d\, i_Y\omega\right) . \tag{4.32}$$

This bracket is antisymmetric, but it does not satisfy the Jacobi identity; the Jacobiator is an exact 1-form. It is, as easily seen, given by a bidifferential operator of total order ≤ 1, so it is a skew quasi algebroid.

The *Dorfman bracket* is defined on the same module of sections. Its definition is the same as for Courant, except that the corrections and the exact part of the second Lie derivative disappear:

$$[X+\omega, Y+\eta] = [X,Y]_{vf} + \mathcal{L}_X\eta - i_Y\,\mathrm{d}\omega = [X,Y]_{vf} + i_X\,\mathrm{d}\eta - i_Y\,\mathrm{d}\omega + \mathrm{d}\, i_X\eta\,. \tag{4.33}$$

This bracket is visibly non skew-symmetric, but it is a Loday bracket which is bidifferential of total order ≤ 1. Moreover, the Dorfman bracket admits the classical left anchor map

$$\rho : \mathcal{T}M = TM \oplus_M T^*M \to TM \tag{4.34}$$

which is the projection onto the first component. Indeed,

$$[X+\omega, f(Y+\eta)] = [X,fY]_{vf} + \mathcal{L}_X f\eta - i_{fY}\,\mathrm{d}\omega = f[X+\omega, Y+\eta] + X(f)(Y+\eta)\,.$$

For the right generalized anchor we have

$$\begin{aligned} [f(X+\omega), Y+\eta] &= [fX,Y]_{vf} + i_{fX}\,\mathrm{d}\eta - i_Y\,\mathrm{d}(f\omega) + \mathrm{d}\, i_{fX}\eta \\ &= f[X+\omega, Y+\eta] - Y(f)(X+\omega) + \mathrm{d}f \wedge (i_X\eta + i_Y\omega)\,, \end{aligned}$$

so that

$$\alpha(Y+\eta)(\mathrm{d}f \otimes (X+\omega)) = \mathrm{d}f \wedge (i_X\eta + i_Y\omega) = 2\langle X+\omega, Y+\eta\rangle_+ \cdot \mathrm{d}f\,,$$

where

$$\langle X+\omega, Y+\eta\rangle_+ = \frac{1}{2}\left(i_X\eta + i_Y\omega\right) = \frac{1}{2}\left(\langle X,\eta\rangle + \langle Y,\omega\rangle\right)\,,$$

is a symmetric nondegenerate bilinear form on $\mathcal{T}M$ (while $\langle\cdot,\cdot\rangle$ is the canonical pairing). We will refer to it, though it is not positively defined, as the *scalar product* in the bundle $\mathcal{T}M$. Note that $\alpha(Y+\eta)$ is really a section of $TM \otimes_M \mathrm{End}(TM \oplus_M T^*M)$ that in local coordinates reads

$$\alpha(Y+\eta) = \sum_k \partial_{x^k} \otimes \left(\mathrm{d}x^k \wedge (i_\eta + i_Y)\right).$$

Hence, the Dorfman bracket is a Loday algebroid bracket.

It is easily checked that the Courant bracket is the antisymmetrization of the Dorfman bracket, and that the Dorfman bracket is the Courant bracket plus $\mathrm{d}\langle X+\omega, Y+\eta\rangle_+$

4.5.3 Twisted Courant-Dorfman bracket

The Courant-Dorfman bracket can be twisted by adding a term associated with a 3-form Θ [KSc05, SW01]:

$$[X+\omega, Y+\eta] = [X,Y]_{vf} + \mathcal{L}_X\eta - i_Y\,\mathrm{d}\omega + i_{X\wedge Y}\Theta\,. \tag{4.35}$$

It turns out that this bracket is still a Loday bracket if the 3-form Θ is closed. As the added term is $C^\infty(M)$-linear with respect to X and Y, the anchors remain the same, thus we deal with a Loday algebroid.

4.5.4 Courant algebroid

Courant algebroids – structures generalizing the Courant-Dorfman bracket on $\mathcal{T}M$ – were introduced as as double objects for Lie bialgebroids by Liu, Weinstein and Xu [LWX97] in a bit complicated way. It was shown by Roytenberg [Roy99] that a Courant algebroid can be equivalently defined as a vector bundle $\tau : E \to M$ with a Loday bracket on $\mathrm{Sec}(E)$, an anchor $\rho : E \to TM$, and a symmetric nondegenerate inner product $(\cdot,\cdot)$ on E, related by a set of four additional properties. It was further observed [Uch02, GM03b] that the number of independent conditions can be reduced.

Definition 4.5.1. A *Courant algebroid* is a vector bundle $\tau : E \to M$ equipped with a Leibniz bracket $[\cdot,\cdot]$ on $\mathrm{Sec}(E)$, a vector bundle map (over the identity) $\rho : E \to TM$, and a nondegenerate symmetric bilinear form (scalar product) $(\cdot|\cdot)$ on E satisfying the identities

$$\rho(X)(Y|Y) = 2(X|[Y,Y]), \tag{4.36}$$

$$\rho(X)(Y|Y) = 2([X,Y]|Y). \tag{4.37}$$

Note that (4.36) is equivalent to

$$\rho(X)(Y|Z) = (X|[Y,Z] + [Z,Y]). \tag{4.38}$$

Similarly, (4.37) easily implies the invariance of the pairing $(\cdot,\cdot)$ with respect to the adjoint maps

$$\rho(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z]), \tag{4.39}$$

which in turn shows that ρ is the anchor map for the left multiplication:

$$[X,fY] = f[X,Y] + \rho(X)(f)Y\,. \tag{4.40}$$

Twisted Courant-Dorfman brackets are examples of Courant algebroid brackets with $(\cdot,\cdot) = \langle\cdot,\cdot\rangle_+$ as the scalar product. Defining a derivation $\mathrm{D} : C^\infty(M) \to \mathrm{Sec}(E)$ by means of the scalar product

$$(\mathrm{D}(f)|X) = \frac{1}{2}\rho(X)(f)\,, \tag{4.41}$$

we get out of (4.38) that

$$[Y,Z] + [Z,Y] = 2\mathrm{D}(Y|Z)\,. \tag{4.42}$$

This, combined with (4.40), implies in turn

$$\alpha(Z)(\mathrm{d}f\otimes Y) = 2(Y|Z)\mathrm{D}(f)\,, \tag{4.43}$$

so any Courant algebroid is a Loday algebroid.

4.5.5 Brackets associated with contact structures

In [Gra13a], contact (super)manifolds have been studied as symplectic principal $\mathbb{R}^\times$-bundles (P, ω); the symplectic form being homogeneous with respect to the $\mathbb{R}^\times$-action. Similarly, Kirillov brackets on line bundles have been regarded as Poisson principal $\mathbb{R}^\times$-bundles. Consequently, *Kirillov algebroids* and *contact Courant algebroids* have been introduced, respectively, as homogeneous Lie algebroids and Courant algebroids on vector bundles equipped with a compatible $\mathbb{R}^\times$-bundle structure. The corresponding brackets are therefore particular Lie algebroid and Courant algebroid brackets, thus Loday algebroid brackets. In other words, Kirillov and contact Courant algebroids are examples of Loday algebroids equipped additionally with some extra geometric structures.
As a canonical example of a contact Courant algebroid, consider the contact 2-manifold represented by the symplectic principal $\mathbb{R}^\times$-bundle $T^*[2]T[1](\mathbb{R}^\times \times M)$, for a purely even manifold M [Gra13a]. As the cubic Hamiltonian H associated with the canonical vector field on $T[1](\mathbb{R}^\times \times M)$ being the de Rham derivative is 1-homogeneous, we obtain a homogeneous Courant bracket on the linear principal $\mathbb{R}^\times$-bundle $P = T(\mathbb{R}^\times \times M) \oplus_{\mathbb{R}^\times \times M} T^*(\mathbb{R}^\times \times M)$. It can be reduced to the vector bundle $E = (\mathbb{R} \times TM) \oplus_M (\mathbb{R}^* \times T^*M)$ whose sections are $(X, f) + (\alpha, g)$, where $f, g \in C^\infty(M)$, X is a vector field, and α is a one-form on M, which is a Loday algebroid bracket of the form

$$[(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)] = ([X_1, X_2]_{vf}, X_1(f_2) - X_2(f_1)) \tag{4.44}$$
$$+ (\mathcal{L}_{X_1}\alpha_2 - i_{X_2}\mathrm{d}\alpha_1 + f_1\alpha_2 - f_2\alpha_1 + f_2\mathrm{d}g_1 + g_2\mathrm{d}f_1, X_1(g_2) - X_2(g_1) + i_{X_2}\alpha_1 + f_1 g_2) \ .$$

This is the Dorfman-like version of the bracket whose skew-symmetrization gives exactly the bracket introduced by Wade [Wa00] to define so called $\mathcal{E}^1(M)$-*Dirac structures* and considered also in [GM03b]. The full contact Courant algebroid structure on E consists additionally [Gra13a] of the symmetric pseudo-Euclidean product

$$\langle (X, f) + (\alpha, g), (X, f) + (\alpha, g) \rangle = \langle X, \alpha \rangle + fg \,,$$

and the vector bundle morphism $\rho^1 : E \to TM \times \mathbb{R}$, corresponding to a map assigning to sections of E first-order differential operators on M, of the form

$$\rho^1\left((X, f) + (\alpha, g)\right) = X + f \ .$$

4.5.6 Grassmann-Dorfman bracket

The Dorfman bracket (4.33) can be immediately generalized to a bracket on sections of $\mathcal{T}^\wedge M = TM \oplus_M \wedge T^*M$, where

$$\wedge T^*M = \bigoplus_{k=0}^{\infty} \wedge^k T^*M \,,$$

so that the module of sections, $\mathrm{Sec}(\wedge T^*M) = \Omega(M) = \bigoplus_{k=0}^\infty \Omega^k(M)$, is the Grassmann algebra of differential forms. The bracket, *Grassmann-Dorfman bracket*, is formally given by the same formula (4.33) and the proof that it is a Loday algebroid bracket is almost the same. The left anchor is the projection on the summand TM,

$$\rho : TM \oplus_M \wedge T^*M \to TM \,, \tag{4.45}$$

and

$$\alpha(Y+\eta)(\mathrm{d}f\otimes(X+\omega)) = \mathrm{d}f\wedge(i_X\eta + i_Y\omega) = 2\,\mathrm{d}f\wedge\langle X+\omega, Y+\eta\rangle_+\,,$$

where

$$\langle X+\omega, Y+\eta\rangle_+ = \frac{1}{2}\left(i_X\eta + i_Y\omega\right)\,,$$

is a symmetric nondegenerate bilinear form on $\mathcal{T}^\wedge M$, this time with values in $\Omega(M)$. Like for the classical Courant-Dorfman bracket, the graph of a differential form β is an isotropic subbundle in $\mathcal{T}^\wedge M$ which is involutive (its sections are closed with respect to the bracket) if and only if $\mathrm{d}\beta = 0$. The Grassmann-Dorfman bracket induces Loday algebroid brackets on all bundles $TM\oplus_M \wedge^k T^*M$, $k = 0, 1, \ldots, \infty$. These brackets have been considered in [BS11] and called there *higher-order Courant brackets* (see also [Zam12]). Note that this is exactly the bracket derived from the bracket of first-order (super)differential operators on the Grassmann algebra $\Omega(M)$: we associate with $X+\omega$ the operator $S_{X+\omega} = i_X + \omega\wedge$ and compute the super-commutators,

$$[[S_{X+\omega}, \mathrm{d}]_{sc}, S_{Y+\eta}]_{sc} = S_{[X+\omega,Y+\eta]}\,.$$

4.5.7 Grassmann-Dorfman bracket for a Lie algebroid

All the above remains valid when we replace TM with a Lie algebroid $(E, [\cdot,\cdot]_E, \rho_E)$, the de Rham differential d with the Lie algebroid cohomology operator d^E on $\mathrm{Sec}(\wedge E^*)$, and the Lie derivative along vector fields with the Lie algebroid Lie derivative $\mathcal{L}^E$. We define a bracket on sections of $E\oplus_M \wedge E^*$ with formally the same formula

$$[X+\omega, Y+\eta] = [X,Y]_E + \mathcal{L}^E_X\eta - i_Y\,\mathrm{d}^E\omega\,. \tag{4.46}$$

This is a Loday algebroid bracket with the left anchor

$$\rho : E\oplus_M \wedge E^* \to TM\,,\quad \rho(X+\omega) = \rho_E(X)$$

and

$$\alpha(Y+\eta)(\mathrm{d}f\otimes(X+\omega)) = \mathrm{d}^E f\wedge(i_X\eta + i_Y\omega)\,.$$

4.5.8 Lie derivative bracket for a Lie algebroid

The above Loday bracket on sections of $E\oplus_M \wedge E^*$ has a simpler version. Let us put simply

$$[X+\omega, Y+\eta] = [X,Y]_E + \mathcal{L}^E_X\eta\,. \tag{4.47}$$

This is again a Loday algebroid bracket with the same left anchor and and

$$\alpha(Y+\eta)(\mathrm{d}f\otimes(X+\omega)) = \mathrm{d}^E f\wedge i_X\eta + \rho_E(Y)(f)\omega\,.$$

In particular, when reducing to 0-forms, we get a Leibniz algebroid structure on $E\times\mathbb{R}$, where the bracket is defined by $[X+f, Y+g] = [X,Y]_E + \rho_E(X)g$, the left anchor by $\rho(X,f) = \rho_E(X)$, and the generalized right anchor by

$$b^r(Y,g)(\mathrm{d}h\otimes(X+f)) = -\rho_E(Y)(h)X\,.$$

In other words,

$$\alpha(Y,g)(\mathrm{d}h\otimes(X+f)) = \rho_E(Y)(h)f\,.$$

4.5.9 Loday algebroids associated with a Nambu-Poisson structure

In the following M denotes a smooth m-dimensional manifold and n is an integer such that $3 \leq n \leq m$. An almost Nambu-Poisson structure of order n on M is an n-linear bracket $\{\cdot, \ldots, \cdot\}$ on $C^\infty(M)$ that is skew-symmetric and has the Leibniz property with respect to the point-wise multiplication. It corresponds to an n-vector field $\Lambda \in \mathrm{Sec}(\wedge^n TM)$. Such a structure is Nambu-Poisson if it verifies the *Filippov identity* (*generalized Jacobi identity*):

$$\begin{gathered}\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \{\{f_1, \ldots, f_{n-1}, g_1\}, g_2, \ldots, g_n\} + \\ \{g_1, \{f_1, \ldots, f_{n-1}, g_2\}, g_3, \ldots, g_n\} + \cdots + \{g_1, \ldots, g_{n-1}, \{f_1, \ldots, f_{n-1}, g_n\}\}\,,\end{gathered} \tag{4.48}$$

i.e., if the Hamiltonian vector fields $X_{f_1 \ldots f_{n-1}} = \{f_1, \ldots, f_{n-1}, \cdot\}$ are derivations of the bracket. Alternatively, an almost Nambu-Poisson structure is Nambu-Poisson if and only if

$$\mathcal{L}_{X_{f_1,\ldots,f_{n-1}}} \Lambda = 0\,,$$

for all functions $f_1, \ldots, f_{n-1}$.

Spaces equipped with skew-symmetric brackets satisfying the above identity have been introduced by Filippov [Fil85] under the name *n-Lie algebras*.

The concept of Leibniz (Loday) algebroid used in [ILMP99] is the usual one, without differentiability condition for the first argument. Actually, this example is a Loday algebroid in our sense as well. The bracket is defined for $(n-1)$-forms by

$$[\omega, \eta] = \mathcal{L}_{\rho(\omega)} \eta + (-1)^n (i_{\mathrm{d}\omega} \Lambda) \eta\,,$$

where

$$\rho : \wedge^{n-1} T^* M \ni \omega \mapsto i_\omega \Lambda \in TM$$

is actually the left anchor. Indeed,

$$[\omega, f\eta] = \mathcal{L}_{\rho(\omega)} f\eta + (-1)^n (i_{\mathrm{d}\omega} \Lambda) f\eta = f[\omega, \eta] + \rho(\omega)(f)\eta\,.$$

For the generalized right anchor we get

$$[f\omega, \eta] = \mathcal{L}_{\rho(f\omega)} \eta + (-1)^n (i_{\mathrm{d}(f\omega)} \Lambda) \eta = f[\omega, \eta] - i_{\rho(\omega)}(\mathrm{d}f \wedge \eta)\,,$$

so

$$\alpha(\eta)(\mathrm{d}f \otimes \omega) = \rho(\eta)(f)\,\omega - \rho(\omega)(f)\,\eta + \mathrm{d}f \wedge i_{\rho(\omega)}\eta\,.$$

Note that α is really a bundle map $\alpha : \wedge^{n-1} T^* M \to TM \otimes_M \mathrm{End}(\wedge^{n-1} T^* M)$, since it is obviously $C^\infty(M)$-linear in η and ω, as well as a derivation with respect to f.

In [Hag02, HM02], another Leibniz algebroid associated with the Nambu-Poisson structure Λ is proposed. The vector bundle is the same, $E = \wedge^{n-1} T^* M$, the left anchor map is the same as well, $\rho(\omega) = i_\omega \Lambda$, but the Loday bracket reads

$$[\omega, \eta]' = \mathcal{L}_{\rho(\omega)} \eta - i_{\rho(\eta)} \mathrm{d}\omega\,.$$

Hence,

$$\begin{aligned}[f\omega, \eta]' &= \mathcal{L}_{\rho(f\omega)} \eta - i_{\rho(\eta)} \mathrm{d}(f\omega) \\ &= f[\omega, \eta]' - \rho(\eta)(f)\,\omega + \mathrm{d}f \wedge (i_{\rho(\omega)}\eta + i_{\rho(\eta)}\omega)\,,\end{aligned}$$

so that for the generalized right anchor we get

$$\alpha(\eta)(\mathrm{d}f\otimes\omega)=\mathrm{d}f\wedge(i_{\rho(\omega)}\eta+i_{\rho(\eta)}\omega)\,.$$

This Loday algebroid structure is clearly the one obtained from the Grassmann-Dorfman bracket on the graph of Λ,

$$\mathrm{graph}(\Lambda)=\{\rho(\omega)+\omega:\omega\in\Omega^{n-1}(M)\}\,.$$

Actually, an n-vector field Λ is a Nambu-Poisson tensor if and only if its graph is closed with respect to the Grassmann-Dorfman bracket [BS11, Hag02].

4.6 The Lie pseudoalgebra of a Loday algebroid

Let us fix a Loday pseudoalgebra bracket $[\cdot,\cdot]$ on an $\mathcal{A}$-module $\mathcal{E}$. Let $\rho:\mathcal{E}\to\mathrm{Der}(\mathcal{A})$ be the left anchor map, and let

$$b^r=\rho-\alpha:\mathcal{E}\to\mathrm{Der}(\mathcal{A})\otimes_{\mathcal{A}}\mathrm{End}(\mathcal{E})$$

be the generalized right anchor map. For every $X\in\mathcal{E}$ we will view $\alpha(X)$ as a $\mathcal{A}$-module homomorphism $\alpha(X):\Omega^1\otimes_{\mathcal{A}}\mathcal{E}\to\mathcal{E}$, where Ω^1 is the $\mathcal{A}$-submodule of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E};\mathcal{A})$ generated by $\mathrm{d}\mathcal{A}=\{\mathrm{d}f:f\in\mathcal{A}\}$ and $\mathrm{d}f(D)=D(f)$.
It is a well-known fact that the subspace $\mathfrak{g}^0$ generated in a Loday algebra $\mathfrak{g}$ by the symmetrized brackets $X\diamond Y=[X,Y]+[Y,X]$ is a two-sided ideal and that $\mathfrak{g}/\mathfrak{g}^0$ is a Lie algebra. Putting

$$\mathcal{E}^0=\mathrm{span}\{[X,X]:X\in\mathcal{E}\}\,,$$

we have then

$$[\mathcal{E}^0,\mathcal{E}]=0\,,\quad[\mathcal{E},\mathcal{E}^0]\subset\mathcal{E}^0\,. \tag{4.49}$$

Indeed, symmetrized brackets are spanned by squares $[X,X]$, so, due to the Jacobi identity,

$$[[X,X],Y]=[X,[X,Y]]-[X,[X,Y]]=0$$

and

$$[Y,[X,X]]=[[Y,X],X]+[X,[Y,X]]=[X,Y]\diamond Y\,. \tag{4.50}$$

However, working with $\mathcal{A}$-modules, we would like to have an $\mathcal{A}$-module structure on $\mathcal{E}/\mathcal{E}^0$. Unfortunately, $\mathcal{E}^0$ is not a submodule in general. Let us consider therefore the $\mathcal{A}$-submodule $\bar{\mathcal{E}}^0$ of $\mathcal{E}$ generated by $\mathcal{E}^0$, i.e., $\bar{\mathcal{E}}^0=\mathcal{A}\cdot\mathcal{E}^0$.

Lemma 4.6.1. *For all $f\in\mathcal{A}$ and $X,Y,Z\in\mathcal{E}$ we have*

$$\alpha(X)(\mathrm{d}f\otimes Y)\ =\ X\diamond(fY)-f(X\diamond Y)\,, \tag{4.51}$$

$$[\alpha(X)(\mathrm{d}f\otimes Y),Z]\ =\ \rho(Z)(f)(X\diamond Y)-\alpha(Z)(\mathrm{d}f\otimes(X\diamond Y))\,. \tag{4.52}$$

In particular,

$$[\alpha(X)(\mathrm{d}f\otimes Y),Z]=[\alpha(Y)(\mathrm{d}f\otimes X),Z]\,. \tag{4.53}$$

Proof. To prove (4.51) it suffices to combine the identity $[X,fY]=f[X,Y]+\rho(X)(f)(Y)$ with

$$[fY,X]=f[Y,X]-\rho(X)(f)Y+\alpha(X)(\mathrm{d}f\otimes Y)\,.$$

Then, as $[\mathcal{E}^0,\mathcal{E}]=0$,

$$[\alpha(X)(\mathrm{d}f\otimes Y),Z]=-[f(X\diamond Y),Z]=\rho(Z)(f)(X\diamond Y)-\alpha(Z)(\mathrm{d}f\otimes(X\diamond Y))\,.$$

□

Corollary 4.6.2. *For all $f \in \mathcal{A}$ and $X, Y \in \mathcal{E}$,*

$$\alpha(X)(\mathrm{d}f \otimes Y) \in \bar{\mathcal{E}}^0\,, \tag{4.54}$$

and the left anchor vanishes on $\bar{\mathcal{E}}^0$,

$$\rho(\bar{\mathcal{E}}^0) = 0\,. \tag{4.55}$$

Moreover, $\bar{\mathcal{E}}^0$ is a two-sided Loday ideal in $\mathcal{E}$ and the Loday bracket induces on the $\mathcal{A}$-module $\bar{\mathcal{E}} = \mathcal{E}/\bar{\mathcal{E}}^0$ a Lie pseudoalgebra structure with the anchor

$$\bar{\rho}([X]) = \rho(X)\,, \tag{4.56}$$

where $[X]$ denotes the coset of X.

Proof. The first statement follows directly from (4.51). As $[\mathcal{E}^0, \mathcal{E}] = 0$, the anchor vanishes on $\mathcal{E}^0$ and thus on $\bar{\mathcal{E}}^0 = \mathcal{A} \cdot \mathcal{E}^0$. From

$$[Z, f(X \diamond Y)] = f[Z, X \diamond Y] + \rho(X)(f)(X \diamond Y) \in \bar{\mathcal{E}}^0$$

and

$$[f(X \diamond Y), Z] = f[(X \diamond Y), Z] - \rho(Z)(f)(X \diamond Y) + \alpha(Z)(\mathrm{d}f \otimes (X \diamond Y)) \in \bar{\mathcal{E}}^0\,,$$

we conclude that $\bar{\mathcal{E}}^0$ is a two-sided ideal. As $\bar{\mathcal{E}}^0$ contains all elements $X \diamond Y$, The Loday bracket induces on $\mathcal{E}/\bar{\mathcal{E}}^0$ a skew-symmetric bracket with the anchor (4.56) and satisfying the Jacobi identity, thus a Lie pseudoalgebra structure. □

Definition 4.6.3. The Lie pseudoalgebra $\bar{\mathcal{E}} = \mathcal{E}/\bar{\mathcal{E}}^0$ we will call the *Lie pseudoalgebra of the Loday pseudoalgebra $\mathcal{E}$*. If $\mathcal{E} = \mathrm{Sec}(E)$ is the Loday pseudoalgebra of a Loday algebroid on a vector bundle E and the module $\bar{\mathcal{E}}^0$ is the module of sections of a vector subbundle $\bar{E}$ of E, we deal with the *Lie algebroid of the Loday algebroid E*.

Example 4.6.4. The Lie algebroid of the Courant-Dorfman bracket is the canonical Lie algebroid TM.

Theorem 4.6.5. *For any Loday pseudoalgebra structure on an $\mathcal{A}$-module $\mathcal{E}$ there is a short exact sequence of morphisms of Loday pseudoalgebras over $\mathcal{A}$,*

$$0 \longrightarrow \bar{\mathcal{E}}^0 \longrightarrow \mathcal{E} \longrightarrow \bar{\mathcal{E}} \longrightarrow 0\,, \tag{4.57}$$

where $\bar{\mathcal{E}}^0$ – the $\mathcal{A}$-submodule in $\mathcal{E}$ generated by $\{[X, X] : X \in \mathcal{E}\}$ – is a Loday pseudoalgebra with the trivial left anchor and $\bar{\mathcal{E}} = \mathcal{E}/\bar{\mathcal{E}}^0$ is a Lie pseudoalgebra.

Note that the Loday ideal $\mathcal{E}^0$ is clearly commutative, while the modular ideal $\bar{\mathcal{E}}^0$ is no longer commutative in general.

4.7 Loday algebroid cohomology

We first recall the definition of the Loday cochain complex associated to a bi-module over a Loday algebra [LP93].

Let $\mathbb{K}$ be a field of nonzero characteristic and V a $\mathbb{K}$-vector space endowed with a (left) Loday bracket $[\cdot, \cdot]$. A *bimodule* over a Loday algebra $(V, [\cdot, \cdot])$ is a $\mathbb{K}$-vector space W together with a left (resp., right) *action* $\mu^l \in \mathrm{Hom}(V \otimes W, W)$ (resp., $\mu^r \in \mathrm{Hom}(W \otimes V, W)$) that verify the following requirements

$$\mu^r[x,y] = \mu^r(y)\mu^r(x) + \mu^l(x)\mu^r(y), \tag{4.58}$$

$$\mu^r[x,y] = \mu^l(x)\mu^r(y) - \mu^r(y)\mu^l(x), \tag{4.59}$$

$$\mu^l[x,y] = \mu^l(x)\mu^l(y) - \mu^l(y)\mu^l(x), \tag{4.60}$$

for all $x, y \in V$.

The *Loday cochain complex* associated to the Loday algebra $(V, [\cdot,\cdot])$ and the bimodule (W, μ^l, μ^r), shortly – to $B = ([\cdot,\cdot], \mu^r, \mu^l)$, is made up by the cochain space

$$\mathrm{Lin}^\bullet(V,W) = \bigoplus_{p\in\mathbb{N}} \mathrm{Lin}^p(V,W) = \bigoplus_{p\in\mathbb{N}} \mathrm{Hom}(V^{\otimes p}, W),$$

where we set $\mathrm{Lin}^0(V,W) = W$, and the coboundary operator ∂_B defined, for any p-cochain c and any vectors $x_1, \ldots, x_{p+1} \in V$, by

$$\begin{aligned}(\partial_B c)(x_1,\ldots,x_{p+1}) \quad = \quad & (-1)^{p+1}\mu^r(x_{p+1})c(x_1,\ldots,x_p) + \sum_{i=1}^{p}(-1)^{i+1}\mu^l(x_i)c(x_1,\ldots\hat{\imath}\ldots,x_{p+1}) \\ & + \sum_{i<j}(-1)^i c(x_1,\ldots\hat{\imath}\ldots,\overbrace{[x_i,x_j]}^{(j)},\ldots,x_{p+1}) \ . \end{aligned} \tag{4.61}$$

Let now ρ be a *representation* of the Loday algebra $(V, [\cdot,\cdot])$ on a $\mathbb{K}$-vector space W, i.e. a Loday algebra homomorphism $\rho : V \to \mathrm{End}(W)$. It is easily checked that $\mu^l := \rho$ and $\mu^r := -\rho$ endow W with a bimodule structure over V. Moreover, in this case of a bimodule induced by a representation, the Loday cohomology operator reads

$$\begin{aligned}(\partial_B c)(x_1,\ldots,x_{p+1}) \quad = \quad & \sum_{i=1}^{p+1}(-1)^{i+1}\rho(x_i)c(x_1,\ldots\hat{\imath}\ldots,x_{p+1}) \\ & + \sum_{i<j}(-1)^i c(x_1,\ldots\hat{\imath}\ldots,\overbrace{[x_i,x_j]}^{(j)},\ldots,x_{p+1}) \ . \end{aligned} \tag{4.62}$$

Note that the above operator ∂_B is well defined if only the *map* $\rho : V \to \mathrm{End}(W)$ and the *bracket* $[\cdot,\cdot] : V \otimes V \to V$ are given. We will refer to it as to the *Loday operator* associated with $B = ([\cdot,\cdot], \rho)$. The point is that $\partial_B^2 = 0$ if and only if $[\cdot,\cdot]$ is a Loday bracket and ρ is its representation. Indeed, the Loday algebra homomorphism property of ρ (resp., the Jacobi identity for $[\cdot,\cdot]$) is encoded in $\partial_B^2 = 0$ on $\mathrm{Lin}^0(V,W) = W$ (resp., $\mathrm{Lin}^1(V,W)$), at least if $W \neq \{0\}$, what we assume).

Let now E be a vector bundle over a manifold M and $B = ([\cdot,\cdot], \rho)$ be an *anchored faint algebroid* structure on E, where $[\cdot,\cdot]$ is a faint pseudoalgebra bracket (bidifferential operator) and $\rho : E \to TM$ is a vector bundle morphism covering the identity, so inducing a module morphism $\rho : \mathrm{Sec}(E) \to \mathrm{Der}(C^\infty(M)) = \mathcal{X}(M)$. It is easy to see that, unlike in the case of a Lie algebroid, the tensor algebra of sections of $\oplus_{k=0}^\infty (E^*)^{\otimes k}$ is, in general, not invariant under the Loday cohomology operator ∂_B associated with $B = ([\cdot,\cdot],\rho)$. Actually, ∂_B rises the degree of a multidifferential operator by one, even when the Loday bracket is skew-symmetric (see e.g. [Ldd98, Ldd04]).

Example 4.7.1. [Ldd04] Suppose that M is a Riemannian manifold with metric tensor g and let ∂_B be the Loday coboundary operator associated with the canonical bracket of vector fields $B = ([\cdot,\cdot]_{\mathrm{vf}}, \mathrm{id}_{TM})$ on $E = TM$. When adopting the conventions of [Ldd04], where the Loday differential associated to right Loday algebras is considered, we then get, for all $X, Y, Z \in \mathcal{X}(M)$,

$$(\partial_B g)(X, Y, Z) = 2g(Y, \nabla_X Z)\,, \tag{4.63}$$

where ∇ is the Levi-Civita connection on M. One can say that the Loday differential of a Riemannian metric defines the corresponding Levi-Civita connection, which clearly is no longer a tensor on M.

The above observation suggests to consider in $\mathrm{Lin}^\bullet(\mathrm{Sec}(E), C^\infty(M))$, instead of $\mathrm{Sec}(\otimes^\bullet E^*)$, the subspace

$$\mathcal{D}^\bullet(\mathrm{Sec}(E), C^\infty(M)) \subset \mathrm{Lin}^\bullet(\mathrm{Sec}(E), C^\infty(M))$$

consisting of all multidifferential operators. If now $B = ([\cdot,\cdot],\rho)$ is an *anchored faint algebroid* structure on E, see above, then it is clear that the space $\mathcal{D}^\bullet(E) := \mathcal{D}^\bullet(\mathrm{Sec}(E), C^\infty(M))$ is stable under the Loday operator ∂_B associated with $B = ([\cdot,\cdot],\rho)$.

In particular, if $([\cdot,\cdot],\rho,\alpha)$ is a Loday algebroid structure on E, its left anchor $\rho : \mathrm{Sec}(E) \to \mathrm{Der}(C^\infty(M)) \subset \mathrm{End}(C^\infty(M))$ is a representation of the Loday algebra $(\mathrm{Sec}(E), [\cdot,\cdot])$ by derivations on $C^\infty(M)$ and $\partial_B^2 = 0$, so ∂_B is a coboundary operator.

Definition 4.7.2. Let $(E, [\cdot,\cdot],\rho,\alpha)$ be a Loday algebroid over a manifold M. We call *Loday algebroid cohomology*, the cohomology of the Loday cochain subcomplex $(\mathcal{D}^\bullet(E), \partial_B)$ associated with $B = ([\cdot,\cdot],\rho)$, i.e. the Loday algebra structure $[\cdot,\cdot]$ on $\mathrm{Sec}(E)$ represented by ρ on $C^\infty(M)$.

4.8 Supercommutative geometric interpretation

Let E be a vector bundle over a manifold M. Looking for a canonical superalgebra structure in $\mathcal{D}^\bullet(E)$, a natural candidate is the *shuffle (super)product*, introduced by Eilenberg and Mac Lane [EML53] (see also [Ree58, Ree60]). It is known that a shuffle algebra on a free associative algebra is a free commutative algebra with the Lyndon words as its free generators [Rad79]. A similar result is valid in the supercommutative case [ZM95]. In this sense the free shuffle superalgebra represents a supercommutative space.

Definition 4.8.1. For any $\ell' \in \mathcal{D}^p(\mathrm{Sec}(E), C^\infty(M))$ and $\ell'' \in \mathcal{D}^q(\mathrm{Sec}(E), C^\infty(M))$, $p, q \in \mathbb{N}$, we define the *shuffle product*

$$(\ell' \pitchfork \ell'')(X_1, \ldots, X_{p+q}) := \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sign}\sigma\ \ell'(X_{\sigma_1}, \ldots, X_{\sigma_p})\, \ell''(X_{\sigma_{p+1}}, \ldots, X_{\sigma_{p+q}}),$$

where the X_i-s denote sections in $\mathrm{Sec}(E)$ and where $\mathrm{Sh}(p,q) \subset \mathbb{S}_{p+q}$ is the subset of the symmetric group $\mathbb{S}_{p+q}$ made up by all (p,q)-shuffles.

The next proposition is well-known.

Proposition 4.8.2. *The space $\mathcal{D}^\bullet(E)$, together with the shuffle multiplication $\pitchfork$, is a graded commutative associative unital $\mathbb{R}$-algebra.*

We refer to this algebra as the *shuffle algebra of the vector bundle $E \to M$*, or simply, of E.

Let $B = ([\cdot,\cdot],\rho)$ be an anchored faint algebroid structure on E and let ∂_B be the associated Loday operator in $\mathcal{D}^\bullet(E)$. Note that we would have $\partial_B^2 = 0$ if we had assumed that we deal with a Loday algebroid.

Denote now by $\mathrm{D}^k(E)$ those k-linear multidifferential operators from $\mathcal{D}^k(E)$ which are of degree 0 with respect to the last variable and of total degree $\leq k-1$, and set $\mathrm{D}^\bullet(E) = \bigoplus_{k=0}^\infty \mathrm{D}^k(E)$. By convention, $\mathrm{D}^0(E) = \mathcal{D}^0(E) = C^\infty(M)$. Moreover, $\mathrm{D}^1(E) = \mathrm{Sec}(E^*)$. It is easy to see that $\mathrm{D}^\bullet(E)$ is stable for the shuffle multiplication. We will call the subalgebra $(\mathrm{D}^\bullet(E), \pitchfork)$, the *reduced shuffle algebra*, and refer to the corresponding graded ringed space as *supercommutative manifold*. Let us emphasize that this denomination is in the present text merely a terminological convention. The graded ringed spaces of the considered type are being investigated in a separate work.

Theorem 4.8.3. *The coboundary operator ∂_B is a degree 1 graded derivation of the shuffle algebra of E, i.e.*

$$\partial(\ell' \pitchfork \ell'') = (\partial\ell') \pitchfork \ell'' + (-1)^p \ell' \pitchfork (\partial\ell''), \tag{4.64}$$

for any $\ell' \in \mathcal{D}^p(E)$ and $\ell'' \in \mathcal{D}^q(E)$. Moreover, if $[\cdot,\cdot]$ is a pseudoalgebra bracket, i.e., if it is of total order ≤ 1 and ρ is the left anchor for $[\cdot,\cdot]$, then ∂_B leaves invariant the reduced shuffle algebra $\mathrm{D}^\bullet(E) \subset \mathcal{D}^\bullet(E)$.

The claim is easily checked on low degree examples. The general proof is as follows.

Proof. The value of the LHS of Equation (4.64) on sections $X_1, \ldots, X_{p+q+1} \in \mathrm{Sec}(E)$ is given by $S_1 + \ldots + S_4$, where

$$S_1 = \sum_{k=1}^{p+1} \sum_{\tau \in \mathrm{Sh}(p,q)} (-1)^{k+1} \mathrm{sign}\tau \; \rho(X_k) \left(\ell'(X_{\tau_1}, \ldots, \widehat{X}_{\tau_k}, \ldots, X_{\tau_{p+1}}) \; \ell''(X_{\tau_{p+2}}, \ldots, X_{\tau_{p+q+1}}) \right)$$

and

$$S_3 = \sum_{1 \leq k < m \leq p+q+1} \sum_{\tau \in \mathrm{Sh}(p,q)} (-1)^k \mathrm{sign}\tau \; \ell'(X_{\tau_1}, \ldots, [X_k, X_m], \ldots) \; \ell''(X_{\tau_-}, \ldots).$$

In the sum S_2, which is similar to S_1, the index k runs through $\{p+2, \ldots, p+q+1\}$ (X_{τ_k} is then missing in ℓ''). The sum S_3 contains those shuffle permutations of $1 \ldots \hat{k} \ldots p+q+1$ that send the argument $[X_k, X_m]$ with index $m =: \tau_r$ into ℓ', whereas S_4 is taken over the shuffle permutations that send $[X_k, X_m]$ into ℓ''.

Analogously, the value of $(\partial\ell') \pitchfork \ell''$ equals $T_1 + T_2$ with

$$T_1 = \sum_{\sigma \in \mathrm{Sh}(p+1,q)} \sum_{i=1}^{p+1} \mathrm{sign}\sigma \, (-1)^{i+1} \left(\rho(X_{\sigma_i}) \, \ell'(X_{\sigma_1}, \ldots, \widehat{X}_{\sigma_i}, \ldots, X_{\sigma_{p+1}}) \right) \ell''(X_{\sigma_{p+2}}, \ldots, X_{\sigma_{p+q+1}})$$

and

$$T_2 = \sum_{\sigma \in \mathrm{Sh}(p+1,q)} \sum_{1 \leq i < j \leq p+1} \mathrm{sign}\sigma \, (-1)^i \, \ell'(X_{\sigma_1}, \ldots, [X_{\sigma_i}, X_{\sigma_j}], \ldots) \, \ell''(X_{\sigma_{p+2}}, \ldots, X_{\sigma_{p+q+1}})$$

(whereas the value $T_3 + T_4$ of $(-1)^p\ \ell' \pitchfork (\partial \ell'')$, which is similar, is not (really) needed in this (sketch of) proof).

Let us stress that in S_3 and T_2 the bracket is in its natural position determined by the index $\tau_r = m$ or σ_j of its second argument, that, since $\mathrm{Sh}(p,q) \simeq \mathbb{S}_{p+q}/(\mathbb{S}_p \times \mathbb{S}_q)$, the number of (p,q)-shuffles equals $(p+q)!/(p!\,q!)$, and that in S_1 the vector field $\rho(X_k)$ acts on a product of functions according to the Leibniz rule, so that each term splits. It is now easily checked that after this splitting the number of different terms in $\rho(X_-)$ (resp. $[X_-, X_-]$) in the LHS and the RHS of Equation (4.64) is equal to $2(p+q+1)!/(p!\,q!)$ (resp. $(p+q)(p+q+1)!/(2\,p!\,q!)$). To prove that both sides coincide, it therefore suffices to show that any term of the LHS can be found in the RHS.

We first check this for any split term of S_1 with vector field action on the value of ℓ' (the proof is similar if the field acts on the second function and also if we choose a split term in S_2),

$$(-1)^{k+1}\mathrm{sign}\tau\ \left(\rho(X_k)\ell'(X_{\tau_1}, \ldots, \widehat{X}_{\tau_k}, \ldots, X_{\tau_{p+1}})\right)\ \ell''(X_{\tau_{p+2}}, \ldots, X_{\tau_{p+q+1}}),$$

where $k \in \{1, \ldots, p+1\}$ is fixed, as well as $\tau \in \mathrm{Sh(p,q)}$ – which permutes $1 \ldots \hat{k} \ldots p+q+1$. This term exists also in T_1. Indeed, the shuffle τ induces a unique shuffle $\sigma \in \mathrm{Sh}(p+1,q)$ and a unique $i \in \{1, \ldots, p+1\}$ such that $\sigma_i = k$. The corresponding term of T_1 then coincides with the chosen term in S_1, since, as easily seen, $\mathrm{sign}\sigma\ (-1)^{i+1} = (-1)^{k+1}\mathrm{sign}\tau$.

Consider now a term in S_3 (the proof is analogous for the terms of S_4),

$$(-1)^k\mathrm{sign}\tau\ \ \ell'(X_{\tau_1}, \ldots, [X_k, X_m], \ldots)\ \ell''(X_{\tau_-}, \ldots),$$

where $k < m$ are fixed in $\{1, \ldots, p+q+1\}$ and where $\tau \in \mathrm{Sh}(p,q)$ is a fixed permutation of $1 \ldots \hat{k} \ldots p+q+1$ such that the section $[X_k, X_m]$ with index $m =: \tau_r$ is an argument of ℓ'. The shuffle τ induces a unique shuffle $\sigma \in \mathrm{Sh}(p+1,q)$. Set $k =: \sigma_i$ and $m =: \sigma_j$. Of course $1 \le i < j \le p+1$. This means that the chosen term reads

$$(-1)^k\mathrm{sign}\tau\ \ \ell'(X_{\sigma_1}, \ldots, [X_{\sigma_i}, X_{\sigma_j}], \ldots, X_{\sigma_{p+1}})\ \ell''(X_{\sigma_{p+2}}, \ldots, X_{\sigma_{p+q+1}}).$$

Finally this term is a term of T_2, as it is again clear that $(-1)^k\mathrm{sign}\tau = \mathrm{sign}\sigma\,(-1)^i$.

That $\mathrm{D}^\bullet(E)$ is invariant under ∂_B in the case of a pseudoalgebra bracket is obvious. This completes the proof. □

Note that the derivations ∂_B of the reduced shuffle algebra (in the case of pseudoalgebra brackets on $\mathrm{Sec}(E)$) are, due to formula (4.62), completely determined by their values on $\mathrm{D}^0(E) \oplus \mathrm{D}^1(E)$. More precisely, $B = ([\cdot,\cdot], \rho)$ can be easily reconstructed from ∂_B thanks to the formulae

$$\rho(X)(f) = \langle X, \partial_B f\rangle \tag{4.65}$$

and

$$\langle \mathfrak{l}, [X,Y]\rangle = \langle X, \partial_B\langle \mathfrak{l}, Y\rangle\rangle - \langle Y, \partial_B\langle \mathfrak{l}, X\rangle\rangle - \partial_B\mathfrak{l}(X,Y)\,, \tag{4.66}$$

where $X, Y \in \mathrm{Sec}(E)$, $\mathfrak{l} \in \mathrm{Sec}(E^*)$, and $f \in C^\infty(M)$.

Theorem 4.8.4. *If ∂ is a derivation of the reduced shuffle algebra $\mathrm{D}^\bullet(E)$, then on $\mathrm{D}^0(E) \oplus \mathrm{D}^1(E)$ the derivation ∂ coincides with ∂_B for a certain uniquely determined $B = ([\cdot,\cdot]_\partial, \rho_\partial)$ associated with a pseudoalgebra bracket $[\cdot,\cdot]_\partial$ on $\mathrm{Sec}(E)$.*

Proof. Let us define $\rho = \rho_\partial$ and $[\cdot,\cdot] = [\cdot,\cdot]_\partial$ out of formulae (4.65) and (4.66), i.e.,

$$\rho(X)(f) = \langle X, \partial f\rangle \tag{4.67}$$

and

$$\langle \mathfrak{l}, [X,Y]\rangle = \langle X, \partial\langle \mathfrak{l}, Y\rangle\rangle - \langle Y, \partial\langle \mathfrak{l}, X\rangle\rangle - \partial\mathfrak{l}(X,Y)\,. \tag{4.68}$$

The fact that $\rho(X)$ is a derivation of $C^\infty(M)$ is a direct consequence of the shuffle algebra derivation property of ∂. Eventually, the map ρ is visibly associated with a bundle map $\rho : E \to TM$.

The bracket $[\cdot,\cdot]$ has ρ as left anchor. Indeed, since $\partial\mathfrak{l}(X,Y)$ is of order 0 with respect to Y, we get from (4.68)

$$[X, fY] - f[X,Y] = \langle X, \partial f\rangle Y = \rho(X)(f)Y\,.$$

Similarly, as $\partial\mathfrak{l}(X,Y)$ is of order 1 with respect to X and of order 0 with respect to Y, the operator

$$\delta_1(f)\,(\partial\mathfrak{l})\,(X,Y) = \partial\mathfrak{l}(fX,Y) - f\partial\mathfrak{l}(X,Y)$$

is $C^\infty(M)$-bilinear, so that the LHS of

$$\langle \mathfrak{l}, [fX,Y] - f[X,Y]\rangle = -\langle Y, \partial f\rangle\langle \mathfrak{l}, X\rangle - \delta_1(f)\,(\partial\mathfrak{l})\,(X,Y),$$

see (4.68), is $C^\infty(M)$-linear with respect to X and Y and a derivation with respect to f. The bracket $[\cdot,\cdot]$ is therefore of total order ≤ 1 with the generalized right anchor $b^r = \rho - \alpha$, where α is determined by the identity

$$\langle \mathfrak{l}, \alpha(Y)(\mathrm{d}f \otimes X)\rangle = \delta_1(f)\,(\partial\mathfrak{l})\,(X,Y)\,. \tag{4.69}$$

This corroborates that α is a bundle map from E to $TM \otimes_M \mathrm{End}(E)$. □

Definition 4.8.5. Let $\mathrm{Der}_1(\mathrm{D}^\bullet(E), ⫛)$ be the space of degree 1 graded derivations ∂ of the reduced shuffle algebra that verify, for any $c \in \mathrm{D}^2(E)$ and any $X_i \in \mathrm{Sec}(E)$, $i = 1, 2, 3$,

$$\begin{aligned}(\partial c)(X_1, X_2, X_3) \;&=\; \sum_{i=1}^{3}(-1)^{i+1}\langle \partial(c(X_1, \ldots \hat{\imath} \ldots, X_3)), X_i\rangle \\ &\quad + \sum_{i<j}(-1)^i\, c(X_1, \ldots \hat{\imath} \ldots, \overset{(j)}{[X_i, X_j]_\partial}, \ldots, X_3)\,.\end{aligned} \tag{4.70}$$

A *homological vector field* of the supercommutative manifold $(M, \mathrm{D}^\bullet(E))$ is a square-zero derivation in $\mathrm{Der}_1(\mathrm{D}^\bullet(E), ⫛)$. Two homological vector fields of $(M, \mathrm{D}^\bullet(E))$ are *equivalent*, if they coincide on $C^\infty(M)$ and on $\mathrm{Sec}(E^*)$.

Observe that Equation (4.70) implies that two equivalent homological fields also coincide on $\mathrm{D}^2(E)$. We are now prepared to give the main theorem of this section.

Theorem 4.8.6. *Let E be a vector bundle. There exists a 1-to-1 correspondence between equivalence classes of homological vector fields*

$$\partial \in \mathrm{Der}_1(\mathrm{D}^\bullet(E), ⫛),\ \partial^2 = 0$$

and Loday algebroid structures on E.

Remark 4.8.7. This theorem is a kind of a non-antisymmetric counterpart of the well-known similar correspondence between homological vector fields of split supermanifolds and Lie algebroids. Furthermore, it may be viewed as an analogue for Loday algebroids of the celebrated Ginzburg-Kapranov correspondence for quadratic Koszul operads [GK94]. According to the latter result, homotopy Loday structures on a graded vector space V correspond bijectively to degree 1 differentials of the Zinbiel algebra $(\bar{\otimes} sV^*, \star)$, where s is the suspension operator and where $\bar{\otimes} sV^*$ denotes the reduced tensor module over sV^*. However, in our geometric setting scalars, or better functions, must be incorporated (see the proof of Theorem 4.8.6), which turns out to be impossible without passing from the Zinbiel multiplication or half shuffle $\star$ to its symmetrization $\pitchfork$. Moreover, it is clear that the algebraic structure on the function sheaf should be associative.

Proof. Let $([\cdot,\cdot],\rho,\alpha)$ be a Loday algebroid structure on the given vector bundle $E \to M$. According to Theorem 4.8.3, the corresponding coboundary operator ∂_B is a square 0 degree 1 graded derivation of the reduced shuffle algebra and (4.70) is satisfied by definition, as $[\cdot,\cdot]_{\partial_B} = [\cdot,\cdot]$.

Conversely, let ∂ be such a homological vector field. According to Theorem 4.8.4, the derivation ∂ coincides on $\mathrm{D}^0(E) \oplus \mathrm{D}^1(E)$ with ∂_B for a certain pseudoalgebra bracket $[\cdot,\cdot] = [\cdot,\cdot]_\partial$ on $\mathrm{Sec}(E)$. Its left anchor is $\rho = \rho_\partial$ and the generalized right anchor $b^r = \rho - \alpha$ is determined by means of formula (4.69), where $\mathfrak{l}$ runs through all sections of E^*.

To prove that the triplet $([\cdot,\cdot],\rho,\alpha)$ defines a Loday algebroid structure on E, it now suffices to check that the Jacobi identity holds true. It follows from (4.68) that

$$\langle \mathfrak{l}, [X_1,[X_2,X_3]]\rangle = -\langle \partial\langle \mathfrak{l}, X_1\rangle, [X_2,X_3]\rangle + \langle \partial\langle \mathfrak{l}, [X_2,X_3]\rangle, X_1\rangle - (\partial\mathfrak{l})(X_1,[X_2,X_3]).$$

Since the first term of the RHS is (up to sign) the evaluation of $[X_2,X_3]$ on the section $\partial\langle \mathfrak{l}, X_1\rangle$ of E^*, and a similar remark is valid for the contraction $\langle \mathfrak{l}, [X_2,X_3]\rangle$ in the second term, we can apply (4.68) also to these two brackets. If we proceed analogously for $[[X_1,X_2],X_3]$ and $[X_2,[X_1,X_3]]$, and use (4.67) and the homological property $\partial^2 = 0$, we find, after simplification, that the sum of the preceding three double brackets equals

$$\sum_{i=1}^{3}(-1)^{i+1}\rho(X_i)(\partial\mathfrak{l})(X_1,\dots\hat{\imath}\dots,X_3) + \sum_{i<j}(-1)^i\,(\partial\mathfrak{l})(X_1,\dots\hat{\imath}\dots,\overbrace{[X_i,X_j]}^{(j)},\dots,X_3)\,.$$

In view of (4.70), the latter expression coincides with $(\partial^2\mathfrak{l})(X_1,X_2,X_3) = 0$, so that the Jacobi identity holds.

It is clear that the just detailed assignment of a Loday algebroid structure to any homological vector field can be viewed as a map on equivalence classes of homological vector fields. □

Having a homological vector field ∂ associated with a Loday algebroid structure $([\cdot,\cdot],\rho,\alpha)$ on E, we can easily develop the corresponding Cartan calculus for the shuffle algebra $\mathcal{D}^\bullet(E)$.

Proposition 4.8.8. *For any $X \in \mathrm{Sec}(E)$, the contraction*

$$\mathcal{D}^p(E) \ni \ell \mapsto i_X\ell \in \mathcal{D}^{p-1}(E)\,, \quad (i_X\ell)(X_1,\dots,X_{p-1}) = \ell(X,X_1,\dots,X_{p-1})\,,$$

is a degree -1 graded derivation of the shuffle algebra $(\mathcal{D}^\bullet(E),\pitchfork)$.

Proof. Using usual notations, our definitions, as well as a separation of the involved shuffles σ into the σ-s that verify $\sigma_1 = 1$ and those for which $\sigma_{p+1} = 1$, we get

$$\left(i_{X_1}(\ell' \pitchfork \ell'')\right)(X_2, \ldots, X_{p+q}) = \sum_{\sigma:\sigma_1=1} \text{sign}\sigma\ (i_{X_1}\ell')(X_{\sigma_2}, \ldots, X_{\sigma_p})\ell''(X_{\sigma_{p+1}}, \ldots, X_{\sigma_{p+q}})$$

$$+ \sum_{\sigma:\sigma_{p+1}=1} \text{sign}\sigma\ \ell'(X_{\sigma_1}, \ldots, X_{\sigma_p})(i_{X_1}\ell'')(X_{\sigma_{p+2}}, \ldots, X_{\sigma_{p+q}}).$$

Whereas a (p,q)-shuffle of the type $\sigma_1 = 1$ is a $(p-1,q)$-shuffle with same signature, a (p,q)-shuffle such that $\sigma_{p+1} = 1$ defines a $(p, q-1)$-shuffle with signature $(-1)^p \text{sign}\sigma$. Therefore, we finally get

$$i_{X_1}(\ell' \pitchfork \ell'') = (i_{X_1}\ell') \pitchfork \ell'' + (-1)^p \ell' \pitchfork (i_{X_1}\ell'').$$

□

Observe that the supercommutators $[i_X, i_Y]_{\text{sc}} = i_X i_Y + i_Y i_X$ do not necessarily vanish, so that the derivations i_X of the shuffle algebra generate a Lie superalgebra of derivations with negative degrees. Indeed, $[i_X, i_Y]_{\text{sc}} =: i_{X\square Y}$, $[[i_X, i_Y]_{\text{sc}}, i_Z]_{\text{sc}} =: i_{(X\square Y)\square Z}$, ... are derivations of degree -2, -3, ... given on any $\ell \in \mathcal{D}^p(E)$ by

$$(i_{X\square Y}\ell)(X_1, \ldots, X_{p-2}) = \ell(Y, X, X_1, \ldots, X_{p-2}) + \ell(X, Y, X_1, \ldots, X_{p-2})\,,$$

$$(i_{(X\square X)\square Y}\ell)(X_1, \ldots, X_{p-3}) = 2\ell(Y, X, X, X_1, \ldots, X_{p-3}) - 2\ell(X, X, Y, X_1, \ldots, X_{p-3})\,, \ldots$$

The next proposition is obvious.

Proposition 4.8.9. *The supercommutator* $\mathcal{L}_X := [\partial, i_X]_{\text{sc}} = \partial i_X + i_X \partial$, $X \in \text{Sec}(E)$, *is a degree 0 graded derivation of the shuffle algebra. Explicitly, for any* $\ell \in \mathcal{D}^p(E)$ *and* $X_1, \ldots, X_p \in \text{Sec}(E)$,

$$(\mathcal{L}_X \ell)(X_1, \ldots, X_p) = \rho(X)\,(\ell(X_1, \ldots, X_p)) - \sum_i \ell(X_1, \ldots, \overbrace{[X, X_i]}^{(i)}, \ldots, X_p)\,. \tag{4.71}$$

We refer to the derivation $\mathcal{L}_X$ as the Loday algebroid *Lie derivative along* X.

If we define the Lie derivative on the tensor algebra $T_{\mathbb{R}}(E) = \bigoplus_{p=0}^{\infty} \text{Sec}(E)^{\otimes_{\mathbb{R}} p}$ in the obvious way by

$$\mathcal{L}_X(X_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} X_p) = \sum_i X_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \overbrace{[X, X_i]}^{(i)} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} X_p\,,$$

and if we use the canonical pairing

$$\langle \ell, X_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} X_p \rangle = \ell(X_1, \ldots, X_p)$$

between $\mathcal{D}^\bullet(E)$ and $T_{\mathbb{R}}(E)$, we get

$$\mathcal{L}_X \langle \ell, X_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} X_p \rangle = \langle \mathcal{L}_X \ell, X_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} X_p \rangle + \langle \ell, \mathcal{L}_X(X_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} X_p) \rangle\,. \tag{4.72}$$

The following theorem is analogous to the results in the standard case of a Lie algebroid $E = TM$ and operations on the Grassmann algebra $\Omega(M) \subset \mathcal{D}^\bullet(TM)$ of differential forms.

Theorem 4.8.10. *The graded derivations ∂, i_X, and $\mathcal{L}_X$ on $\mathcal{D}^\bullet(E)$ satisfy the following identities:*

(a) $2\partial^2 = [\partial, \partial]_{\mathrm{sc}} = 0$,

(b) $\mathcal{L}_X = [\partial, i_X]_{\mathrm{sc}} = \partial i_X + i_X \partial$,

(c) $\partial \mathcal{L}_X - \mathcal{L}_X \partial = [\partial, \mathcal{L}_X]_{\mathrm{sc}} = 0$,

(d) $\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = [\mathcal{L}_X, i_Y]_{\mathrm{sc}} = i_{[X,Y]}$,

(e) $\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = [\mathcal{L}_X, \mathcal{L}_Y]_{\mathrm{sc}} = \mathcal{L}_{[X,Y]}$.

Proof. The results (a), (b), and (c) are obvious. Identity (d) is immediately checked by direct computation. The last equality is a consequence of (c), (d), and the Jacobi identity applied to $[\mathcal{L}_X, [\partial, i_Y]_{\mathrm{sc}}]_{\mathrm{sc}}$. □

Note that we can easily calculate the Lie derivatives of negative degrees, $\mathcal{L}_{X \Box Y} := [\partial, i_{X \Box Y}]_{\mathrm{sc}}$, $\mathcal{L}_{(X \Box Y) \Box Z} := [\partial, i_{(X \Box Y) \Box Z}]_{\mathrm{sc}}$, ... with the help of the graded Jacobi identity.

Observe finally that Item (d) of the preceding theorem actually means that

$$i_{[X,Y]} = [\![i_X, i_Y]\!]_\partial,$$

where the RHS is the restriction to interior products of the derived bracket on $\mathrm{Der}(\mathcal{D}^\bullet(E), \pitchfork)$ defined by the graded Lie bracket $[\cdot, \cdot]_{\mathrm{sc}}$ and the interior Lie algebra derivation $[\partial, \cdot]_{\mathrm{sc}}$ of $\mathrm{Der}(\mathcal{D}^\bullet(E), \pitchfork)$ induced by the homological vector field ∂.

5 On the infinity category of homotopy Leibniz algebras

The following research work is a cooperation with Prof. Dr. Norbert Poncin and Dr. Jian Qiu, which is being published in the ArXiv preprint database and submitted for publication in a peer-reviewed international journal.

5.1 Introduction

5.1.1 General background

Homotopy, sh, or infinity algebras [Sta63] are homotopy invariant extensions of differential graded algebras. They are of importance, e.g. in BRST of closed string field theory, in Deformation Quantization of Poisson manifolds ... Another technique to increase the flexibility of algebraic structures is categorification [CF94], [Cra95] – a sharpened viewpoint that leads to astonishing results in TFT, bosonic string theory ... Both methods, homotopification and categorificiation are tightly related: the 2-categories of 2-term Lie (resp., Leibniz) infinity algebras and of Lie (resp., Leibniz) 2-algebras turned out to be equivalent [BC04], [SL10] (for a comparison of 3-term Lie infinity algebras and Lie 3-algebras, as well as for the categorical definition of the latter, see [KMP11]). However, homotopies of ∞-morphisms and their compositions are far from being fully understood. In [BC04], ∞-homotopies are obtained from categorical homotopies, which are God-given. In [SS07], (higher) ∞-homotopies are (higher) derivation homotopies, a variant of infinitesimal concordances, which seems to be the wrong concept [DP12]. In [Sho08], the author states that ∞-homotopies of sh Lie algebra morphisms can be composed, but no proof is given and the result is actually not true in whole generality. The objective of this work is to clarify the concept of (higher) ∞-homotopies, as well as the problem of their compositions.

5.1.2 Structure and main results

In Section 5.2, we provide explicit formulae for Leibniz infinity algebras and their morphisms. Indeed, although a category of homotopy algebras is simplest described as a category of quasi-free DG coalgebras, its original nature is its manifestation in terms of brackets and component maps.

We report, in Section 5.3, on the notions of homotopy that are relevant for our purposes: concordances, i.e. homotopies for morphisms between quasi-free DG (co)algebras, gauge and Quillen homotopies for Maurer-Cartan (MC for short) elements of pronilpotent Lie infinity algebras, and ∞-homotopies, i.e. gauge or Quillen homotopies for ∞-morphisms viewed as MC elements of a complete convolution Lie infinity algebra.

Section 5.4 starts with the observation that vertical composition of ∞-homotopies of DG algebras is well-defined. However, this composition is not associative and cannot be extended to the ∞-algebra case – which suggests that ∞-algebras actually form an ∞-category. To allow independent reading of the present paper, we provide a short introduction to ∞-categories, see Subsection 5.4.2. In Subsection 5.4.3.1, the concept of ∞-n-homotopy is made precise and the class of ∞-algebras is viewed as an ∞-category. Since we apply the proof of the Kan property of the nerve of a nilpotent Lie infinity algebra to the 2-term Leibniz infinity case, a good understanding of this proof is indispensable: we detail the latter in Subsection 5.4.3.2.

To be complete, we give an explicit description of the category of 2-term Leibniz infinity algebras at the beginning of Section 5.5. We show that composition of ∞-homotopies in the nerve-∞-groupoid, which is defined and associative only up to higher ∞-homotopy, projects to a well-defined and associative vertical composition in the 2-term case – thus obtaining the Leibniz counterpart of the strict 2-category of 2-term Lie infinity algebras [BC04], see Subsection 5.5.2, Theorem 5.5.5 and Theorem 5.5.7.

Eventually, we provide, in Section 5.6, the definitions of the strict 2-category of Leibniz 2-algebras, which is 2-equivalent to the preceding 2-category.

An ∞-category structure on the class of ∞-algebras over a quadratic Koszul operad is being investigated independently of [Get09] in a separate paper.

5.2 Category of Leibniz infinity algebras

Let P be a quadratic Koszul operad. Surprisingly enough, P_∞-structures on a graded vector space V (over a field $\mathbb{K}$ of characteristic zero), which are essentially sequences ℓ_n of n-ary brackets on V that verify a sequence R_n of defining relations, $n \in \{1, 2, \ldots\}$, are 1:1 [GK94] with codifferentials

$$D \in \mathrm{CoDer}^1(\mathcal{F}^{\mathrm{gr,c}}_{P^{\mathrm{i}}}(s^{-1}V)) \quad (|\ell_n| = 2 - n) \quad \text{or} \quad D \in \mathrm{CoDer}^{-1}(\mathcal{F}^{\mathrm{gr,c}}_{P^{\mathrm{i}}}(sV)) \quad (|\ell_n| = n - 2)\ , \tag{5.1}$$

or, also, (if V is finite-dimensional) 1:1 with differentials

$$d \in \mathrm{Der}^1(\mathcal{F}^{\mathrm{gr}}_{P^!}(sV^*)) \quad (|\ell_n| = 2 - n) \quad \text{or} \quad d \in \mathrm{Der}^{-1}(\mathcal{F}^{\mathrm{gr}}_{P^!}(s^{-1}V^*)) \quad (|\ell_n| = n - 2)\ . \tag{5.2}$$

Here $\mathrm{Der}^1(\mathcal{F}^{\mathrm{gr}}_{P^!}(sV^*))$ (resp., $\mathrm{CoDer}^1(\mathcal{F}^{\mathrm{gr,c}}_{P^{\mathrm{i}}}(s^{-1}V))$), for instance, denotes the space of endomorphisms of the free graded algebra over the Koszul dual operad $P^!$ of P on the suspended linear dual sV^* of V, which have degree 1 (with respect to the grading of the free algebra that is induced by the grading of V) and are derivations for each binary operation in $P^!$ (resp., the space of endomorphisms of the free graded coalgebra over the Koszul dual cooperad P^{i} on the desuspended space $s^{-1}V$ that are coderivations) (by differential and codifferential we mean of course a derivation or coderivation that squares to 0).

Although the original nature of homotopified or oidified algebraic objects is their manifestation in terms of brackets [BP12], the preceding coalgebraic and algebraic settings are the most convenient contexts to think about such higher structures.

5.2.1 Zinbiel (co)algebras

Since we take an interest mainly in the case where P is the operad Lei (resp., the operad Lie) of Leibniz (resp., Lie) algebras, the Koszul dual $P^!$ to consider is the operad Zin (resp., Com) of Zinbiel (resp., commutative) algebras. We now recall the relevant definitions and results.

Definition 5.2.1. A *graded Zinbiel algebra* (*GZA*) (resp., *graded Zinbiel coalgebra* (*GZC*)) *is a $\mathbb{Z}$-graded vector space V endowed with a multiplication, i.e. a degree 0 linear map $m : V \otimes V \to V$ (resp., a comultiplication, i.e. a degree 0 linear map $\Delta : V \to V \otimes V$) that verifies the relation*

$$m(\mathrm{id}\otimes m) = m(m\otimes\mathrm{id}) + m(m\otimes\mathrm{id})(\tau\otimes\mathrm{id}) \quad (\textit{resp.,}\ (\mathrm{id}\otimes\Delta)\Delta = (\Delta\otimes\mathrm{id})\Delta + (\tau\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\Delta)\ , \tag{5.3}$$

where $\tau : V \otimes V \ni u \otimes v \mapsto (-1)^{|u||v|} v \otimes u \in V \otimes V$.

Of course, when evaluated on homogeneous vectors $u, v, w \in V$, the Zinbiel relation for the multiplication $m(u, v) =: u \cdot v$ reads,

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w + (-1)^{|u||v|} (v \cdot u) \cdot w \; .$$

Example 5.2.2. The multiplication $\cdot$ on the reduced tensor module $\overline{T}(V) := \oplus_{n \geq 1} V^{\otimes n}$ over a $\mathbb{Z}$-graded vector space V, defined, for homogeneous $v_i \in V$, by

$$\begin{aligned} &(v_1 ... v_p) \cdot (v_{p+1} ... v_{p+q}) = \sum_{\sigma \in \mathrm{Sh}(p, q-1)} (\sigma^{-1} \otimes \mathrm{id})(v_1 ... v_{p+q}) = \\ &= \sum_{\sigma \in \mathrm{Sh}(p, q-1)} \varepsilon(\sigma^{-1}) v_{\sigma^{-1}(1)} v_{\sigma^{-1}(2)} ... v_{\sigma^{-1}(p+q-1)} v_{p+q} \; , \end{aligned} \tag{5.4}$$

where we wrote tensor products of vectors by simple juxtaposition, where $\mathrm{Sh}(p, q-1)$ is the set of $(p, q-1)$-shuffles, and where $\varepsilon(\sigma^{-1})$ is the Koszul sign, endows $\overline{T}(V)$ with a *GZA* structure.

Similarly, the comultiplication Δ on $\overline{T}(V)$, defined, for homogeneous $v_i \in V$, by

$$\Delta(v_1 ... v_p) = \sum_{k=1}^{p-1} \sum_{\sigma \in \mathrm{Sh}(k, p-k-1)} \varepsilon(\sigma) \left(v_{\sigma(1)} ... v_{\sigma(k)}\right) \bigotimes \left(v_{\sigma(k+1)} ... v_{\sigma(p-k-1)} v_p\right) \; , \tag{5.5}$$

is a *GZC* structure on $\overline{T}(V)$.

As for the GZA multiplication on $\overline{T}(V)$, we have in particular

$$v_1 \cdot v_2 = v_1 v_2 \; ; \quad (v_1 v_2) \cdot v_3 = v_1 v_2 v_3 \; ;$$

$$v_1 \cdot (v_2 v_3) = v_1 v_2 v_3 + (-1)^{|v_1||v_2|} v_2 v_1 v_3; \quad (((v_1 \cdot v_2) \cdot v_3) ...) \cdot v_k = v_1 v_2 ... v_k \; .$$

Proposition 5.2.3. *The* GZA $(\overline{T}(V), \cdot)$ (*resp., the* GZC $(\overline{T}(V), \Delta)$) *defined in Example 5.2.2 is the* free GZA (*resp.,* free GZC) *over* V*. We will denote it by* $\mathrm{Zin}(V)$ (*resp.,* $\mathrm{Zin}^{\mathrm{c}}(V)$)*.*

Definition 5.2.4. A *differential graded Zinbiel algebra* (*DGZA*) (resp., a *differential graded Zinbiel coalgebra*) (*DGZC*) is a *GZA* (V, m) (resp., *GZC* (V, Δ)) together with a degree 1 (-1 in the homological setting) derivation d (resp., coderivation D) that squares to 0. More precisely, d (resp., D) is a degree 1 (-1 in the homological setting) linear map $d : V \to V$ (resp., $D : V \to V$), such that

$$d\, m = m\, (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \quad (\text{resp.,} \quad \Delta\, D = (D \otimes \mathrm{id} + \mathrm{id} \otimes D)\, \Delta)$$

and $d^2 = 0$ (resp., $D^2 = 0$).

Since the GZA $\mathrm{Zin}(V)$ (resp., GZC $\mathrm{Zin}^{\mathrm{c}}(V)$) is free, any degree 1 linear map $d : V \to \mathrm{Zin}(V)$ (resp., $D : \mathrm{Zin}^{\mathrm{c}}(V) \to V$) uniquely extends to a derivation $d : \mathrm{Zin}(V) \to \mathrm{Zin}(V)$ (resp., coderivation $D : \mathrm{Zin}^{\mathrm{c}}(V) \to \mathrm{Zin}^{\mathrm{c}}(V)$).

Definition 5.2.5. A *quasi-free* DGZA (resp., a *quasi-free* DGZC) over V is a DGZA (resp., DGZC) of the type $(\mathrm{Zin}(V), d)$ (resp., $(\mathrm{Zin}^{\mathrm{c}}(V), D)$).

5.2.2 Leibniz infinity algebras

In the present text we use homological (i-ary map of degree $i-2$) and cohomological (i-ary map of degree $2-i$) infinity algebras. Let us recall the definition of homological Leibniz infinity algebras.

Definition 5.2.6. A (homological) *Leibniz infinity algebra* is a graded vector space V together with a sequence of linear maps $l_i : V^{\otimes i} \to V$ of degree $i-2$, $i \geq 1$, such that for any $n \geq 1$, the following *higher Jacobi identity* holds:

$$\sum_{i+j=n+1} \sum_{j\leqslant k\leqslant n} \sum_{\sigma\in \mathrm{Sh}(k-j,j-1)} (-1)^{(n-k+1)(j-1)} (-1)^{j(v_{\sigma(1)}+...+v_{\sigma(k-j)})} \varepsilon(\sigma)\,\mathrm{sign}(\sigma)$$
$$l_i(v_{\sigma(1)},...,v_{\sigma(k-j)}, l_j(v_{\sigma(k-j+1)},...,v_{\sigma(k-1)},v_k), v_{k+1},...,v_n) = 0\ , \tag{5.6}$$

where signσ is the signature of σ and where we denoted the degree of the homogeneous $v_i \in V$ by v_i instead of $|v_i|$.

Theorem 5.2.7. *There is a 1:1 correspondence between Leibniz infinity algebras, in the sense of Definition 5.2.6, over a graded vector space V and quasi-free* DGZC*-s* $(\mathrm{Zin}^{\mathrm{c}}(sV), D)$ *(resp., in the case of a finite-dimensional graded vector space V, quasi-free* DGZA*-s* $(\mathrm{Zin}(s^{-1}V^*), d)$*).*

In the abovementioned 1:1 correspondence between infinity algebras over a quadratic Koszul operad P and quasi-free DG$P^{¡}$C (resp., quasi-free DG$P^{!}$A) (self-explaining notation), a P_∞-algebra structure on a graded vector space V is viewed as a representation on V of the DG operad P_∞ – which is defined as the cobar construction $\Omega P^{¡}$ of the Koszul dual cooperad $P^{¡}$. Theorem 5.2.7 makes this correspondence concrete in the case $P = \mathsf{Lei}$; a proof can be found in [AP10] and in the Appendix of this thesis (see page 133).

5.2.3 Leibniz infinity morphisms

Definition 5.2.8. A *morphism between Leibniz infinity algebras* (V, l_i) and (W, m_i) is a sequence of linear maps $\varphi_i : V^{\otimes i} \to W$ of degree $i-1$, $i \geq 1$, which satisfy, for any $n \geq 1$, the condition

$$\sum_{i=1}^{n} \sum_{k_1+...+k_i=n} \sum_{\sigma\in\mathfrak{Sh}(k_1,...,k_i)} (-1)^{\sum_{r=1}^{i-1}(i-r)k_r+\frac{i(i-1)}{2}} (-1)^{\sum_{r=2}^{i}(k_r-1)(v_{\sigma(1)}+...+v_{\sigma(k_1+...+k_{r-1})})} \varepsilon(\sigma)\,\mathrm{sign}(\sigma)$$
$$m_i\left(\varphi_{k_1}(v_{\sigma(1)},...,v_{\sigma(k_1)}), \varphi_{k_2}(v_{\sigma(k_1+1)},...,v_{\sigma(k_1+k_2)}),...,\varphi_{k_i}(v_{\sigma(k_1+...+k_{i-1}+1)},...,v_{\sigma(k_1+...+k_i)})\right)$$
$$=$$
$$\sum_{i+j=n+1} \sum_{j\leqslant k\leqslant n} \sum_{\sigma\in \mathrm{Sh}(k-j,j-1)} (-1)^{k+(n-k+1)j} (-1)^{j(v_{\sigma(1)}+...+v_{\sigma(k-j)})} \varepsilon(\sigma)\,\mathrm{sign}(\sigma)$$
$$\varphi_i(v_{\sigma(1)},...,v_{\sigma(k-j)}, l_j(v_{\sigma(k-j+1)},...,v_{\sigma(k-1)},v_k), v_{k+1},...,v_n)\ , \tag{5.7}$$

where $\mathfrak{Sh}(k_1,\ldots,k_i)$ denotes the set of shuffles $\sigma \in \mathrm{Sh}(k_1,\ldots,k_i)$, such that $\sigma(k_1) < \sigma(k_1+k_2) < \ldots < \sigma(k_1+k_2+\ldots+k_i)$.

Theorem 5.2.9. *There is a 1:1 correspondence between Leibniz infinity algebra morphisms from* (V, l_i) *to* (W, m_i) *and* DGC *morphisms* $\mathrm{Zin}^{\mathrm{c}}(sV) \to \mathrm{Zin}^{\mathrm{c}}(sW)$ *(resp., in the finite-dimensional* *case,* DGA *morphisms* $\text{Zin}(s^{-1}W^*) \to \text{Zin}(s^{-1}V^*)$), *where the quasi-free* DGZC-*s* (*resp., the quasi-free* DGZA-*s*) *are endowed with the codifferentials* (*resp., differentials*) *that encode the structure maps* l_i *and* m_i.

In literature, infinity morphisms of P_∞-algebras are usually defined as morphisms of quasi-free DG$P^{¡}$C-s. However, no explicit formulae seem to exist for the Leibniz case. A proof of Theorem 5.2.9 can be found in the Appendix (see page 135). Let us also stress that the concept of infinity morphism of P_∞-algebras does not coincide with the notion of morphism of algebras over the operad P_∞.

5.2.4 Composition of Leibniz infinity morphisms

Composition of infinity morphisms between P_∞-algebras corresponds to composition of the corresponding morphisms between quasi-free DG$P^{¡}$C-s: the categories P_∞-`Alg` and `qfDG`$P^{¡}$`CoAlg` (self-explaining notation) are isomorphic. Explicit formulae can easily be computed.

5.3 Leibniz infinity homotopies

5.3.1 Concordances and their compositions

Let us first look for a proper concept of homotopy in the category `qfDG`$P^{¡}$`CoAlg`, or, dually, in `qfDG`$P^{!}$`Alg`.

5.3.1.1 Definition and characterization The following concept of homotopy – referred to as concordance – first appeared in an unpublished work by Stasheff and Schlessinger, which was based on ideas of Bousfield and Gugenheim. It can also be found in [SSS07], for homotopy algebras over the operad `Lie` (algebraic version), as well as in [DP12], for homotopy algebras over an arbitrary operad P (coalgebraic version).

It is well-known that a C^∞-homotopy $\eta : I \times X \to Y$, $I = [0,1]$, connecting two smooth maps p, q between two smooth manifolds X, Y, induces a cochain homotopy between the pullbacks p^*, q^*. Indeed, in the algebraic category,

$$\eta^* : \Omega(Y) \to \Omega(I) \otimes \Omega(X) \ ,$$

and $\eta^*(\omega)$, $\omega \in \Omega(Y)$, reads

$$\eta^*(\omega)(t) = \varphi(\omega)(t) + dt\, \rho(\omega)(t) \ . \tag{5.8}$$

It is easily checked (see below for a similar computation) that, since η^* is a cochain map, we have

$$d_t\varphi = d_X\rho(t) + \rho(t)d_Y \ ,$$

where d_X, d_Y are the de Rham differentials. When integrating over I, we thus obtain

$$q^* - p^* = d_X h + h d_Y \ ,$$

where $h = \int_I \rho(t)dt$ has degree -1.

Before developing a similar approach to homotopies between morphisms of quasi-free DGZA-s, let us recall that tensoring an 'algebra' (resp., 'coalgebra') with a DGCA (resp., DGCC) does not change the considered type of algebra (resp., coalgebra); let us also introduce the 'evaluation' maps

$$\varepsilon_1^i : \Omega(I) = C^\infty(I) \oplus dt\, C^\infty(I) \ni f(t) + dt\, g(t) \mapsto f(i) \in \mathbb{K}, \quad i \in \{0,1\} .$$

In the following – in contrast with our above notation – we omit stars. Moreover – although the 'algebraic' counterpart of a Leibniz infinity algebra over V is $(\mathrm{Zin}(s^{-1}V^*), d_V)$ – we consider Zinbiel algebras of the type $(\mathrm{Zin}(V), d_V)$.

Definition 5.3.1. If $p, q : \mathrm{Zin}(W) \to \mathrm{Zin}(V)$ are two *DGA* morphisms, a *homotopy* or *concordance* $\eta : p \Rightarrow q$ from p to q is a *DGA* morphism $\eta : \mathrm{Zin}(W) \to \Omega(I) \otimes \mathrm{Zin}(V)$, such that

$$\varepsilon_1^0 \eta = p \quad \text{and} \quad \varepsilon_1^1 \eta = q .$$

The following proposition is basic.

Proposition 5.3.2. *Concordances*

$$\eta : \mathrm{Zin}(W) \to \Omega(I) \otimes \mathrm{Zin}(V)$$

between DGA *morphisms p, q can be identified with 1-parameter families*

$$\varphi : I \to \mathrm{Hom}_{\mathrm{DGA}}(\mathrm{Zin}(W), \mathrm{Zin}(V))$$

and

$$\rho : I \to \varphi\mathrm{Der}(\mathrm{Zin}(W), \mathrm{Zin}(V))$$

of (degree 0) DGA *morphisms and of degree* 1 *φ-Leibniz morphisms, respectively, such that*

$$d_t \varphi = [d, \rho(t)] \tag{5.9}$$

and $\varphi(0) = p$, $\varphi(1) = q$. The RHS *of the differential equation (5.9) is defined by*

$$[d, \rho(t)] := d_V \rho(t) + \rho(t) d_W ,$$

where d_V, d_W are the differentials of the quasi-free DGZA*-s* $\mathrm{Zin}(V), \mathrm{Zin}(W)$.

The notion of φ-derivation or φ-Leibniz morphism appeared for instance in [BKS04]: for $w, w' \in \mathrm{Zin}(W)$, w homogeneous,

$$\rho(w \cdot w') = \rho(w) \cdot \varphi(w') + (-1)^w \varphi(w) \cdot \rho(w') ,$$

where we omitted the dependence of ρ on t.

Proof. As already mentioned in Equation (5.8), $\eta(w)$, $w \in \mathrm{Zin}(W)$, reads

$$\eta(w)(t) = \varphi(w)(t) + dt\, \rho(w)(t) ,$$

where $\varphi(t) : \mathrm{Zin}(W) \to \mathrm{Zin}(V)$ and $\rho(t) : \mathrm{Zin}(W) \to \mathrm{Zin}(V)$ have degrees 0 and 1, respectively (the grading of $\mathrm{Zin}(V)$ is induced by that of V and the grading of $\Omega(I)$ is the homological one). Let us now translate the remaining properties of η into properties of φ and ρ. We denote by $d_I = dt\, d_t$ the de Rham differential of I. Since η is a chain map,

$$dt\, d_t \varphi + d_V \varphi - dt\, d_V \rho = (d_I \otimes \mathrm{id} + \mathrm{id} \otimes d_V)\eta = \eta d_W = \varphi d_W + dt\, \rho d_W ,$$

so that

$$d_V\varphi = \varphi d_W \quad \text{and} \quad d_t\varphi = d_V\rho + \rho d_W = [d, \rho] \ .$$

As η is also an algebra morphism, we have, for $w, w' \in \mathrm{Zin}(W)$,

$$\varphi(w \cdot w') + dt\, \rho(w \cdot w') = (\varphi(w) + dt\, \rho(w)) \cdot (\varphi(w') + dt\, \rho(w'))$$

$$= \varphi(w) \cdot \varphi(w') + (-1)^w dt\, (\varphi(w) \cdot \rho(w')) + dt\, (\rho(w) \cdot \varphi(w')) \ ,$$

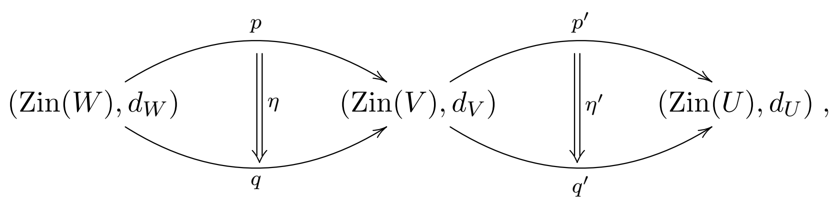
and φ (resp., ρ) is a family of DGA morphisms (resp., of degree 1 φ-Leibniz maps) from $\mathrm{Zin}(W)$ to $\mathrm{Zin}(V)$. Eventually,

$$p = \varepsilon_1^0 \eta = \varphi(0) \quad \text{and} \quad q = \varepsilon_1^1 \eta = \varphi(1) \ .$$

□

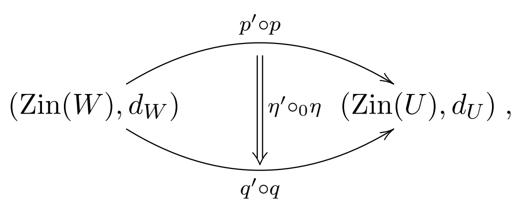
5.3.1.2 Horizontal and vertical compositions *In literature, the 'categories' of Leibniz (resp., Lie) infinity algebras over V (finite-dimensional) and of quasi-free DGZA-s (resp., quasi-free DGCA-s) over* $s^{-1}V^*$ *are (implicitly or explicitly) considered equivalent.* This conjecture is so far corroborated by the results of this paper. Hence, let us briefly report on compositions of concordances.

Let $\eta : p \Rightarrow q$ and $\eta' : p' \Rightarrow q'$,



(5.10)

be concordances between DGA morphisms. Their horizontal composite $\eta' \circ_0 \eta : p' \circ p \Rightarrow q' \circ q$,

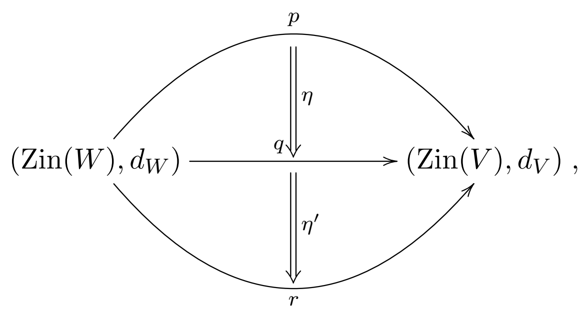


is defined by

$$(\eta' \circ_0 \eta)(t) = (\varphi'(t) \circ \varphi(t)) + dt\, (\varphi'(t) \circ \rho(t) + \rho'(t) \circ \varphi(t)) \ , \qquad (5.11)$$

with self-explaining notation. It is easily checked that the first term of the RHS and the coefficient of dt in the second term have the properties needed to make $\eta' \circ_0 \eta$ a concordance between $p' \circ p$ and $q' \circ q$.

As for the vertical composite $\eta' \circ_1 \eta : p \Rightarrow r$ of concordances $\eta : p \Rightarrow q$ and $\eta' : q \Rightarrow r$,



note that the composability condition $\varphi(1) = q = t(\eta) = s(\eta') = q = \varphi'(0)$, where s, t denote the source and target maps, does not encode any information about $\rho(1), \rho'(0)$. Hence, the usual 'half-time' composition cannot be applied.

Remark 5.3.3. The preceding observation is actually the shadow of the fact that the 'category' of Leibniz infinity algebras is an infinity category.

5.3.2 Infinity homotopies

Some authors addressed directly or indirectly the concept of homotopy of Lie infinity algebras (L_∞-algebras). As aforementioned, in the (equivalent) 'category' of quasi-free DGCA-s, the classical picture of homotopy leads to concordances. In the 'category' of L_∞-algebras itself, morphisms can be viewed as Maurer-Cartan (MC) elements of a specific L_∞-algebra [Dol07],[Sho08], which yields the notion of 'gauge homotopy' between L_∞-morphisms. Additional notions of homotopy between MC elements do exist: Quillen and cylinder homotopies. On the other hand, Markl [Mar02] uses colored operads to construct homotopies for ∞-morphisms in a systematic way. The concepts of concordance, operadic homotopy, as well as Quillen, gauge, and cylinder homotopies are studied in detail in [DP12], for homotopy algebras over any Koszul operad, and they are shown to be equivalent, essentially due to homotopy transfer.

In this subsection, we focus on the Leibniz infinity case and provide a brief account on the relationship between concordances, gauge homotopies, and Quillen homotopies (in the next section, we explain why the latter concept is the bridge to Getzler's [Get09] (and Henriques' [Hen08]) work, as well as to the infinity category structure on the set of Leibniz infinity algebras).

Let us stress that all series in this section converge under some local finiteness or nilpotency conditions (for instance pronilpotency or completeness).

5.3.2.1 Gauge homotopic Maurer-Cartan elements Lie infinity algebras over $\mathfrak{g}$ are in bijective correspondence with quasi-free DGCC-s $(\mathrm{Com}^c(s\mathfrak{g}), D)$, see Equation (5.1). Depending on the definition of the i-ary brackets ℓ_i, $i \geq 1$, from the corestrictions $D_i : (s\mathfrak{g})^{\odot i} \to s\mathfrak{g}$, where $\odot$ denotes the graded symmetric tensor product, one obtains various sign conventions in the defining relations of a Lie infinity algebra. When setting $\ell_i := D_i$ (resp., $\ell_i := s^{-1} D_i\, s^{\,i}$ (our choice in this paper), $\ell_i := (-1)^{i(i-1)/2} s^{-1} D_i\, s^{\,i}$), we get a Voronov L_∞-antialgebra [Vor05] (resp., a Lada-Stasheff L_∞-algebra [LS93], a Getzler L_∞-algebra [Get09]) made up by graded symmetric multilinear maps $\ell_i : (s\mathfrak{g})^{\times i} \to s\mathfrak{g}$ of degree -1 (resp., by graded antisymmetric multilinear maps $\ell_i : \mathfrak{g}^{\times i} \to \mathfrak{g}$ of degree $i-2$, idem), which verify the conditions

$$\sum_{i+j=r+1} \sum_{\sigma \in \mathrm{Sh}(i,j-1)} \varepsilon(\sigma) \ell_j(\ell_i(sv_{\sigma_1}, \ldots, sv_{\sigma_i}), sv_{\sigma_{i+1}}, \ldots, sv_{\sigma_r}) = 0 , \tag{5.12}$$

for all homogeneous $sv_k \in s\mathfrak{g}$ and all $r \geq 1$ (resp., the same higher Jacobi identities (5.12), except that the sign $\varepsilon(\sigma)$ is replaced by $(-1)^{i(j-1)}\varepsilon(\sigma)\mathrm{sign}(\sigma)$, by $(-1)^i\varepsilon(\sigma)\mathrm{sign}(\sigma)$).

As the MC equation of a Lie infinity algebra $(\mathfrak{g}, \ell_i)$ must correspond to the MC equation given by the D_i, it depends on the definition of the operations ℓ_i. For a Lada-Stasheff L_∞-algebra (resp., a Getzler L_∞-algebra), we obtain that the set $\mathrm{MC}(\mathfrak{g})$ of MC elements of $\mathfrak{g}$ is the set of solutions $\alpha \in \mathfrak{g}_{-1}$ of the MC equation

$$\sum_{i=1}^{\infty} \frac{1}{i!}(-1)^{i(i-1)/2} \ell_i(\alpha, \ldots, \alpha) = 0 \quad (\text{resp.,} \sum_{i=1}^{\infty} \frac{1}{i!} \ell_i(\alpha, \ldots, \alpha) = 0) . \tag{5.13}$$

Remark 5.3.4. Since we prefer the original definition of homotopy Lie [LS93] and homotopy Leibniz [AP10] algebras, but use the results of [Get09], we adopt – to facilitate the comparison with [Get09] – Getzler's sign convention whenever his work is involved, and change signs appropriately later, when applying the results in our context.

Hence, we now consider the second MC equation (5.13). Further, for any $\alpha \in \mathfrak{g}_{-1}$, the twisted brackets

$$\ell_i^\alpha(v_1, \ldots, v_i) = \sum_{k=0}^{\infty} \frac{1}{k!} \ell_{k+i}(\alpha^{\otimes k}, v_1, \ldots, v_i) ,$$

$v_1, \ldots, v_i \in \mathfrak{g}$, are a sequence of graded antisymmetric multilinear maps of degree $i-2$. It is well-known that the ℓ_i^α endow $\mathfrak{g}$ with a new Lie infinity structure, if $\alpha \in \mathrm{MC}(\mathfrak{g})$. Finally, any vector $r \in \mathfrak{g}_0$ gives rise to a vector field

$$V_r : \mathfrak{g}_{-1} \ni \alpha \mapsto V_r(\alpha) := -\ell_1^\alpha(r) = -\sum_{k=0}^{\infty} \frac{1}{k!} \ell_{k+1}(\alpha^{\otimes k}, r) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} \ell_k(r, \alpha^{\otimes (k-1)}) \in \mathfrak{g}_{-1} . \tag{5.14}$$

This field restricts to a vector field of the Maurer-Cartan quadric $\mathrm{MC}(\mathfrak{g})$ [DP12]. It follows that the integral curves

$$d_t\alpha = V_r(\alpha(t)) , \tag{5.15}$$

starting from points in the quadric, are located inside $\mathrm{MC}(\mathfrak{g})$. Hence, the

Definition 5.3.5. ([Dol07], [Sho08]) Two MC elements $\alpha, \beta \in \mathrm{MC}(\mathfrak{g})$ of a Lie infinity algebra $\mathfrak{g}$ are *gauge homotopic* if there exists $r \in \mathfrak{g}_0$ and an integral curve $\alpha(t)$ of V_r, such that $\alpha(0) = \alpha$ and $\alpha(1) = \beta$.

This gauge action is used to define the deformation functor $\mathrm{Def} : \mathtt{L}_\infty \to \mathtt{Set}$ from the category of Lie infinity algebras to the category of sets. Moreover, it will provide a concept of homotopy between Leibniz infinity morphisms.

Let us first observe that Equation (5.15) is a 1-variable ordinary differential equation (ODE) and can be solved via an iteration procedure. The integral curve with initial point $\alpha \in \mathrm{MC}(\mathfrak{g})$ is computed in [Get09]. When using our sign convention in the defining relations of a Lie infinity algebra, we get an ODE that contains different signs and the solution of the corresponding Cauchy problem reads

$$\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} e^k_\alpha(r) \;, \tag{5.16}$$

where the $e^k_\alpha(r)$ admit a nice combinatorial description in terms of rooted trees. Moreover, they can be obtained inductively:

$$\begin{cases} e^{i+1}_\alpha(r) = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^{\frac{n(n+1)}{2}} \sum_{k_1+...+k_n=i} \frac{i!}{k_1!...k_n!} \ell_{n+1}(e^{k_1}_\alpha(r), ..., e^{k_n}_\alpha(r), r) \;, \\ e^0_\alpha(r) = \alpha \;. \end{cases} \tag{5.17}$$

It follows that $\alpha, \beta \in \mathrm{MC}(\mathfrak{g})$ are gauge homotopic if

$$\beta - \alpha = \sum_{k=1}^{\infty} \frac{1}{k!} e^k_\alpha(r) \;, \tag{5.18}$$

for some $r \in \mathfrak{g}_0$.

5.3.2.2 Simplicial de Rham algebra We first fix the notation.

Let Δ be the *simplicial category* with objects the nonempty finite ordinals $[n] = \{0, \ldots, n\}$, $n \geq 0$, and morphisms the order-respecting functions $f : [m] \to [n]$. Denote by $\delta^i_n : [n-1] \rightarrowtail [n]$ the injection that omits the image i and by $\sigma^i_n : [n+1] \twoheadrightarrow [n]$ the surjection that assigns the same image to i and $i+1$, $i \in \{0, \ldots, n\}$.

A *simplicial object* in a category $\mathtt{C}$ is a functor $X \in [\Delta^{\mathrm{op}}, \mathtt{C}\,]$. It is completely determined by the simplicial data $(X_n, d^{\,n}_{\,i}, s^{\,n}_{\,i})$, $n \geq 0$, $i \in \{0, \ldots, n\}$, where $X_n = X[n]$ (n-simplices), $d^{\,n}_{\,i} = X(\delta^i_n)$ (face maps), and $s^{\,n}_{\,i} = X(\sigma^i_n)$ (degeneracy maps). We denote by $\mathtt{SC}$ the functor category $[\Delta^{\mathrm{op}}, \mathtt{C}]$ of simplicial objects in $\mathtt{C}$.

The simplicial category is embedded in its Yoneda dual category:

$$h_* : \Delta \ni [n] \mapsto \mathrm{Hom}_\Delta(-, [n]) \in [\Delta^{\mathrm{op}}, \mathtt{Set}] = \mathtt{SSet} \;.$$

We refer to the functor of points of $[n]$, i.e. to the simplicial set $\Delta[n] := \mathrm{Hom}_\Delta(-, [n])$, as the *standard simplicial n-simplex*. Moreover, the Yoneda lemma states that

$$\mathrm{Hom}_\Delta([n], [m]) \simeq \mathrm{Hom}(\mathrm{Hom}_\Delta(-, [n]), \mathrm{Hom}_\Delta(-, [m])) = \mathrm{Hom}(\Delta[n], \Delta[m]) \;.$$

This bijection sends $f : [n] \to [m]$ to φ defined by $\varphi_{[k]}(\bullet) = f \circ \bullet$ and φ to $\varphi_{[n]}(\mathrm{id}_{[n]})$. In the following we identify $[n]$ (resp., f) with $\Delta[n]$ (resp., φ).

The set S_n of n-simplices of a simplicial set S is obviously given by $S_n \simeq \mathrm{Hom}(\mathrm{Hom}_\Delta(-, [n]), S) = \mathrm{Hom}(\Delta[n], S)$.

Let us also recall the adjunction

$$|-| : \mathtt{SSet} \rightleftarrows \mathtt{Top} : \mathrm{Sing}$$

given by the 'geometric realization functor' $|-|$ and the 'singular complex functor' Sing. To define $|-|$, we first define the realization $|\Delta[n]|$ of the standard simplicial n-simplex to be the *standard topological n-simplex*

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, \sum_i x_i = 1\} \;.$$

We can view $|-|$ as a functor $|-| \in [\Delta, \mathtt{Top}]$. Indeed, if $f:[n]\to[m]$ is an order-respecting map, we can define a continuous map $|f|:\Delta^n\to\Delta^m$ by

$$|f|[x_0,\ldots,x_n]=[y_0,\ldots,y_m]\ ,$$

where $y_i=\sum_{j\in f^{-1}\{i\}} x_j$.

Let $\wedge^\star_{\mathbb{K}}(x_0,\ldots,x_n,dx_0,\ldots,dx_n)$ be the free graded commutative algebra generated over $\mathbb{K}$ by the degree 0 (resp., degree 1) generators x_i (resp., dx_i). If we divide the relations $\sum_i x_i=1$ and $\sum_i dx_i=0$ out and set $d(x_i)=dx_i$ and $d(dx_i)=0$, we obtain a quotient DGCA

$$\Omega^\star_n=\wedge^\star_{\mathbb{K}}(x_0,\ldots,x_n,dx_0,\ldots,dx_n)/(\textstyle\sum_i x_i-1,\sum_i dx_i)$$

that can be identified, for $\mathbb{K}=\mathbb{R}$, with the algebra of polynomial differential forms $\Omega^\star(\Delta^n)$ of the standard topological n-simplex Δ^n. When defining $\Omega^\star:\Delta^{\mathrm{op}}\to\mathtt{DGCA}$ by $\Omega^\star[n]:=\Omega^\star_n$ and, for $f:[n]\to[m]$, by $\Omega^\star(f):=|f|^*:\Omega^\star_m\to\Omega^\star_n$ (use the standard pullback formula for differential forms given by $y_i=\sum_{j\in f^{-1}\{i\}} x_j$), we obtain a simplicial differential graded commutative algebra $\Omega^\star\in\mathtt{SDGCA}$. Hence, the face maps $d^{\,n}_i:\Omega^\star_n\to\Omega^\star_{n-1}$ are the pullbacks by the $|\delta^i_n|:\Delta^{n-1}\to\Delta^n$, and similarly for the degeneracy maps. In particular, $d^2_0=|\delta^0_2|^*:\Omega^\star_2\to\Omega^\star_1$ is induced by $y_0=0,y_1=x_0,y_2=x_1$. Let us eventually introduce, for $0\le i\le n$, the vertex e_i of Δ^n whose $(i+1)$-th component is equal to 1, as well as the evaluation map $\varepsilon^i_n:\Omega^\star_n\to\mathbb{K}$ at e_i ('pullback' induced by $(y_0,\ldots,y_n)=e_i$).

5.3.2.3 Quillen homotopic Maurer-Cartan elements We already mentioned that, if $(\mathfrak{g},\ell_i)$ is an L_∞-algebra and $(A,\cdot,d)$ a DGCA, their tensor product $\mathfrak{g}\otimes A$ has a canonical L_∞-structure $\bar\ell_i$. It is given by

$$\bar\ell_1(v\otimes a)=(\ell_1\otimes\mathrm{id}+\mathrm{id}\otimes d)(v\otimes a)=\ell_1(v)\otimes a+(-1)^v v\otimes d(a)$$

and, for $i\ge 2$, by

$$\bar\ell_i(v_1\otimes a_1,\ldots,v_i\otimes a_i)=\pm\ell_i(v_1,\ldots,v_i)\otimes(a_1\cdot\ldots\cdot a_i)\ ,$$

where $\pm$ is the Koszul sign generated by the commutation of the variables.

The following concept originates in Rational Homotopy Theory.

Definition 5.3.6. Two MC elements $\alpha,\beta\in\mathrm{MC}(\mathfrak{g})$ of a Lie infinity algebra $\mathfrak{g}$ are *Quillen homotopic* if there exists a MC element $\bar\gamma\in\mathrm{MC}(\bar{\mathfrak{g}}_1)$ of the Lie infinity algebra $\bar{\mathfrak{g}}_1:=\mathfrak{g}\otimes\Omega^\star_1$, such that $\varepsilon^0_1\bar\gamma=\alpha$ and $\varepsilon^1_1\bar\gamma=\beta$ (where the ε^i_1 are the natural extensions of the evaluation maps).

From now on, we accept, in the definition of gauge equivalent MC elements, vector fields $V_{r(t)}$ induced by time-dependent $r=r(t)\in\mathfrak{g}_0$. The next result is proved in [Can99] (see also [Man99]). A proof sketch will be given later.

Proposition 5.3.7. *Two MC elements of a Lie infinity algebra are Quillen homotopic if and only if they are gauge homotopic.*

5.3.2.4 Infinity morphisms as Maurer-Cartan elements and infinity homotopies

The possibility to view morphisms in $\mathrm{Hom}_{\mathrm{DGP^iC}}(C, \mathcal{F}^{\mathrm{gr,c}}_{P^{\mathrm{i}}}(sW))$ as MC elements is known from the theory of the bar and cobar constructions of algebras over an operad. In [DP12], the authors show that the fact that infinity morphisms between P_∞-algebras V and W, i.e. morphisms in

$$\mathrm{Hom}_{\mathrm{DGP^iC}}(\mathcal{F}^{\mathrm{gr,c}}_{P^{\mathrm{i}}}(sV), \mathcal{F}^{\mathrm{gr,c}}_{P^{\mathrm{i}}}(sW)) ,$$

are 1:1 with Maurer-Cartan elements of an L_∞-structure on

$$\mathrm{Hom}_{\mathbb{K}}(\mathcal{F}^{\mathrm{gr,c}}_{P^{\mathrm{i}}}(sV), W) ,$$

is actually a consequence of a more general result based on the encoding of two P_∞-algebras and an infinity morphism between them in a DG colored free operad. In the case $P = \mathsf{Lie}$, one recovers the fact [Sho08] that

$$\mathrm{Hom}_{\mathrm{DGCC}}(C, \mathrm{Com}^c(sW)) \simeq \mathrm{MC}(\mathrm{Hom}_{\mathbb{K}}(C, W)) , \tag{5.19}$$

where C is any locally conilpotent DGCC, where W is an L_∞-algebra, and where the RHS is the set of MC elements of some convolution L_∞-structure on $\mathrm{Hom}_{\mathbb{K}}(C, W)$.

In the sequel we detail the case $P = \mathsf{Lei}$. Indeed, when interpreting infinity morphisms of Leibniz infinity algebras as MC elements of a Lie infinity algebra, the equivalent notions of gauge and Quillen homotopies provide a concept of homotopy between Leibniz infinity morphisms.

Proposition 5.3.8. *Let (V, ℓ_i) and (W, m_i) be two Leibniz infinity algebras and let $(\mathrm{Zin}^c(sV), D)$ be the quasi-free DGZC that corresponds to (V, ℓ_i). The graded vector space*

$$L(V, W) := \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(sV), W)$$

carries a convolution Lie infinity structure given by

$$\mathcal{L}_1 f = m_1 \circ f + (-1)^f f \circ D \tag{5.20}$$

and, as for $\mathcal{L}_p(f_1, ..., f_p)$, $p \geq 2$, by

$$\mathrm{Zin}^c(sV) \xrightarrow{\Delta^{p-1}} (\mathrm{Zin}^c(sV))^{\otimes p} \xrightarrow{\sum_{\sigma \in S(p)} \varepsilon(\sigma)\, \mathrm{sign}(\sigma)\, f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(p)}} W^{\otimes p} \xrightarrow{m_p} W , \tag{5.21}$$

where $f, f_1, \ldots, f_p \in L(V, W)$, $\Delta^{p-1} = (\Delta \otimes \mathrm{id}^{\otimes(p-2)}) \ldots (\Delta \otimes \mathrm{id})\Delta$, where $S(p)$ denotes the symmetric group on p symbols, and where the central arrow is the graded antisymmetrization operator.

Proof. See [Laa02] or [DP12]. A direct verification is possible as well. □

Proposition 5.3.9. *Let (V, ℓ_i) and (W, m_i) be two Leibniz infinity algebras. There exists a 1:1 correspondence between the set of infinity morphisms from (V, ℓ_i) to (W, m_i) and the set of MC elements of the convolution Lie infinity algebra structure $\mathcal{L}_i$ on $L(V, W)$ defined in Proposition 5.3.8.*

Observe that the considered MC series converges pointwise. Indeed, the evaluation of $\mathcal{L}_p(f_1,...,f_p)$ on a tensor in $\mathrm{Zin}^c(sV)$ vanishes for $p \gg$, in view of the local conilpotency of $\mathrm{Zin}^c(sV)$. Moreover, convolution L_∞-algebras are complete, so that their MC equation converges in the topology induced by the filtration (a descending filtration F^iL of the space L of an L_∞-algebra $(L,\mathcal{L}_k)$ is *compatible* with the L_∞-structure $\mathcal{L}_k$, if $\mathcal{L}_k(F^{i_1}L,\ldots,F^{i_k}L)\subset F^{i_1+\ldots+i_k}L$, and it is *complete*, if, in addition, the 'universal' map $L\to \varprojlim L/F^iL$ from L to the (projective) limit of the inverse system L/F^iL is an isomorphism) [DP12].

Note also that Proposition 5.3.9 is a specification, in the case $P=\mathsf{Lei}$, of the abovementioned 1:1 correspondence between infinity morphisms of P_∞-algebras and MC elements of a convolution L_∞-algebra. To increase the readability of this text, we give nevertheless a sketchy proof.

Proof. An MC element is an $\alpha\in\mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(sV),W)$ of degree -1 that verifies the MC equation. Hence, $s\alpha:\mathrm{Zin}^c(sV)\to sW$ has degree 0 and, since $\mathrm{Zin}^c(sW)$ is free as GZC, $s\alpha$ coextends uniquely to $\widehat{s\alpha}\in\mathrm{Hom}_{\mathrm{GZC}}(\mathrm{Zin}^c(sV),\mathrm{Zin}^c(sW))$. The fact that α is a solution of the MC equation exactly means that $\widehat{s\alpha}$ is a DGZC-morphism, i.e. an infinity morphism between the Leibniz infinity algebras V and W. Indeed, when using e.g. the relations $\ell_i=s^{-1}D_i\,s^i$ and $m_i=s^{-1}\mathfrak{D}_i\,s^i$, and the corresponding version of the MC equation, we get

$$\sum_{p=1}^{\infty}\tfrac{1}{p!}(-1)^{\frac{p(p-1)}{2}}\mathcal{L}_p(\alpha,...,\alpha)=0\Leftrightarrow$$
$$\sum_{p=1}^{\infty}(-1)^{\frac{p(p-1)}{2}}m_p(\alpha\otimes...\otimes\alpha)\Delta^{p-1}+(-1)^{\alpha}\alpha D=0\Leftrightarrow$$
$$\sum_{p=1}^{\infty}(-1)^{\frac{p(p-1)}{2}}s^{-1}\mathfrak{D}_p\,s^p(\alpha\otimes...\otimes\alpha)\Delta^{p-1}+(-1)^{\alpha}\alpha D=0\Leftrightarrow$$
$$\sum_{p=1}^{\infty}s^{-1}\mathfrak{D}_p(s\alpha\otimes...\otimes s\alpha)\Delta^{p-1}-s^{-1}s\alpha D=0\Leftrightarrow$$
$$\mathfrak{D}\widehat{(s\alpha)}=\widehat{(s\alpha)}D\ .$$

□

Hence, the

Definition 5.3.10. Two infinity morphisms f,g between Leibniz infinity algebras (V,ℓ_i), (W,m_i) are *infinity homotopic*, if the corresponding MC elements $\alpha=\alpha(f)$ and $\beta=\beta(g)$ of the convolution Lie infinity structure $\mathcal{L}_i$ on $L=L(V,W)$ are Quillen (or gauge) homotopic. In other words, f and g are infinity homotopic, if there exists $\bar\gamma\in\mathrm{MC}_1(\bar L)$, i.e. an MC element $\bar\gamma$ of the Lie infinity structure $\bar{\mathcal{L}}_i$ on $\bar L=L\otimes\Omega_1^\star$ – obtained from the convolution structure $\mathcal{L}_i$ on $L=\mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(sV),W)$ via extension of scalars –, such that $\varepsilon_1^0\bar\gamma=\alpha$ and $\varepsilon_1^1\bar\gamma=\beta$.

5.3.2.5 Comparison of concordances and infinity homotopies Since, according to the prevalent philosophy, the 'categories' `qfDG`$P^{¡}$`CoAlg` and P_∞`-Alg` are 'equivalent', appropriate concepts of homotopy in both categories should be in 1:1 correspondence. It can be shown [DP12] that, for any type of algebras, the concepts of concordance and of Quillen homotopy are equivalent (at least if one defines concordances in an appropriate way); as, Quillen homotopies as already known to be equivalent to gauge homotopies, the wanted result follows in whole generality.

To accommodate the reader who is not interested in (nice) technicalities, we provide now a sketchy explanation of both relationships, defining concordances dually and assuming for simplicity that $\mathbb{K} = \mathbb{R}$.

Remember first that we defined concordances, in conformity with the classical picture, in a contravariant way: two infinity morphisms $f, g : V \to W$ between homotopy Leibniz algebras, i.e. two DGA morphisms $f^*, g^* : \mathrm{Zin}(s^{-1}W^*) \to \mathrm{Zin}(s^{-1}V^*)$, are concordant if there is a morphism

$$\eta \in \mathrm{Hom}_{\mathrm{DGA}}(\mathrm{Zin}(s^{-1}W^*), \mathrm{Zin}(s^{-1}V^*) \otimes \Omega_1^\star) ,$$

whose values at 0 and 1 are equal to f^* and g^*, respectively. Although we will use this definition in the sequel (observe that we slightly adapted it to future purposes), we temporarily prefer a dual, covariant definition (which has the advantage that the spaces V, W need not be finite-dimensional).

The problem that the linear dual of the infinite-dimensional DGCA $\Omega_1^\star$ (let us recall that $\star$ stands for the (co)homological degree) is not a coalgebra, has already been addressed in [BM12]. The authors consider a space $(\Omega_1^\star)^\vee$ made up by the formal series, with coefficients in $\mathbb{K}$, of the elements $\alpha_i = (t^i)^\vee$, $i \geq 0$, and $\beta_i = (t^i\, dt)^\vee$, $i \geq 0$. For instance, $\sum_{i\in\mathbb{N}} \mathfrak{K}^i \alpha_i$ represents the map $\{t^i\}_{i\in\mathbb{N}} \to \mathbb{K}$ and assigns to each t^i the coefficient $\mathfrak{K}^i$. The differential ∂ of $(\Omega_1^\star)^\vee$ is (dual to the de Rham differential d and is) defined by $\partial(\alpha_i) = 0$ and $\partial(\beta_i) = (i+1)\alpha_{i+1}$. As for the coalgebra structure δ, we set

$$\delta(\alpha_i) = \sum_{a+b=i} \alpha_a \otimes \alpha_b ,$$
$$\delta(\beta_i) = \sum_{a+b=i} (\beta_a \otimes \alpha_b + \alpha_a \otimes \beta_b) .$$

When extending to all formal series, we obtain a map $\delta : (\Omega_1^\star)^\vee \to (\Omega_1^\star)^\vee \widehat{\otimes} (\Omega_1^\star)^\vee$, whose target is the completed tensor product. To fix this difficulty, one considers the decreasing sequence of vector spaces $(\Omega_1^\star)^\vee =: \Lambda^0 \supset \Lambda^1 \supset \Lambda^2 \supset \dots$, where $\Lambda^i = \delta^{-1}(\Lambda^{i-1} \otimes \Lambda^{i-1})$, $i \geq 1$, and defines the universal DGCC $\Lambda := \cap_{i\geq 1} \Lambda^i$.

A concordance can then be defined as a map

$$\eta \in \mathrm{Hom}_{\mathrm{DGC}}(\mathrm{Zin}^c(sV) \otimes \Lambda, \mathrm{Zin}^c(sW))$$

(with the appropriate boundary values). It is then easily seen that any Quillen homotopy, i.e. any element in $\mathrm{MC}(L \otimes \Omega_1^\star)$, gives rise to a concordance. Indeed, when writing $\mathcal{V}$ instead of sV, we have maps

$$L \otimes \Omega_1^\star = \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(\mathcal{V}), W) \otimes \Omega_1^\star = \mathrm{Hom}_{\mathbb{K}}\left(\bigoplus_{i\geq 1} \mathcal{V}^{\otimes i}, W\right) \otimes \Omega_1^\star = \left(\prod_{i\geq 1} \mathrm{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i}, W)\right) \otimes \Omega_1^\star$$
$$\longrightarrow \prod_{i\geq 1} \left(\mathrm{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i}, W) \otimes \Omega_1^\star\right) \longrightarrow \prod_{i\geq 1} \mathrm{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i} \otimes \Lambda, W) = \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(\mathcal{V}) \otimes \Lambda, W) .$$

Only the second arrow is not entirely obvious. Let $(f, \omega) \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i}, W) \times \Omega_1^\star$ and let $(v, \lambda) \in \mathcal{V}^{\otimes i} \times \Lambda$. Set $\omega = \sum_{a=0}^N k_a t^a + \sum_{b=0}^N \kappa_b\,(t^b\, dt)$ and, for instance, $\lambda = \sum_{i\in\mathbb{N}} \mathfrak{K}^i \alpha_i$. Then

$$g(f, \omega)(v, \lambda) := (\sum_{i\in\mathbb{N}} \mathfrak{K}^i \alpha_i)(\omega) f(v) := (\sum_{a=0}^N \mathfrak{K}^a k_a) f(v) \in W$$

defines a map between the mentioned spaces. Eventually, a degree -1 element of $L\otimes\Omega_1^\star$ (resp., an MC element of $L\otimes\Omega_1^\star$, a Quillen homotopy) is sent to an element of $\mathrm{Hom}_{\mathrm{GC}}(\mathrm{Zin}^c(\mathcal{V})\otimes\Lambda, \mathrm{Zin}^c\mathcal{W})$ (resp., an element of $\mathrm{Hom}_{\mathrm{DGC}}(\mathrm{Zin}^c(\mathcal{V})\otimes\Lambda, \mathrm{Zin}^c\mathcal{W})$, a concordance).

The relationship between Quillen and gauge homotopy is (at least on the chosen level of rigor) much clearer. Indeed, an element $\bar\gamma \in \mathrm{MC}_1(\bar L) = \mathrm{MC}(L\otimes\Omega_1^\star)$ can be decomposed as

$$\bar\gamma = \gamma(t)\otimes 1 + r(t)\otimes dt,$$

where $t\in[0,1]$ is the coordinate of Δ^1. When unraveling the MC equation of the $\bar{\mathcal{L}}_i$ according to the powers of dt, one gets

$$\begin{aligned} &\textstyle\sum_{p=1}^\infty \frac{1}{p!}\mathcal{L}_p(\gamma(t),...,\gamma(t)) = 0\ ,\\ &\textstyle\frac{d\gamma}{dt} = -\sum_{p=0}^\infty \frac{1}{p!}\mathcal{L}_{p+1}(\gamma(t),...,\gamma(t),r(t))\ . \end{aligned} \tag{5.22}$$

A (nonobvious) direct computation allows to see that the latter ODE, see Definition 5.3.5 of gauge homotopies and Equations (5.15) and (5.14), is dual (up to dimensional issues) to the ODE (5.9), see Proposition 5.3.2 that characterizes concordances.

5.4 Infinity category of Leibniz infinity algebras

We already observed that vertical composition of concordances is not well-defined and that Leibniz infinity algebras should form an infinity category. It is instructive to first briefly look at infinity homotopies between infinity morphisms of DG algebras.

5.4.1 DG case

Remember that infinity homotopies can be viewed as integral curves of specific vector fields V_r of the MC quadric (with obvious endpoints). In the DG case, we have, for any $r\in L_0$,

$$V_r : L_{-1}\ni\alpha\mapsto V_r(\alpha) = -\mathcal{L}_1(r) - \mathcal{L}_2(\alpha,r)\in L_{-1}\ .$$

In view of the Campbell-Baker-Hausdorff formula,

$$\exp(tV_r)\circ\exp(tV_s) = \exp(tV_r + tV_s + 1/2\ t^2[V_r,V_s] + ...)\ .$$

The point is that

$$V : L_0 \to \mathrm{Vect}(L_{-1})$$

is a Lie algebra morphism – also after restriction to the MC quadric; we will not detail this nonobvious fact. It follows that

$$\exp(tV_r)\circ\exp(tV_s) = \exp(tV_{r+s+1/2\ t[r,s]+...})\ .$$

If we accept, as mentioned previously, time-dependent r-s, the problem of the vertical composition of homotopies is solved in the considered DG situation: the integral curve of the composed homotopy of two homotopies $\exp(tV_s)$ (resp., $\exp(tV_r)$) between morphisms f,g (resp., g,h) is given by

$$c(t) = (\exp(tV_r) \circ \exp(tV_s))(f) = \exp(tV_{r+s+1/2\ t[r,s]+\dots})(f)\ .$$

Note that this vertical composition is not associative. Moreover, the preceding approach does not go through in the infinity situation (note e.g. that in this case L_0 is no longer a Lie algebra). This again points out that homotopy algebras form infinity categories.

5.4.2 Shortcut to infinity categories

This subsection is a short digression that should allow to grasp the spirit of infinity categories. For additional information, we refer the reader to [Gro10],[Fin11] and [Nog12].

Strict n-categories or *strict ω-categories* (in the sense of strict infinity categories) are well understood, see e.g. [KMP11]. Roughly, they are made up by 0-morphisms (objects), 1-morphisms (morphisms between objects), 2-morphisms (homotopies between morphisms)..., up to n-morphisms, except in the ω-case, where this upper bound does not exist. All these morphisms can be composed in various ways, the compositions being associative, admitting identities, etc. However, in most cases of higher categories these defining relations do not hold strictly. A number of concepts of weak infinity category, e.g. infinity categories in which the structural relations hold up to coherent higher homotopy, are developed in literature. Moreover, an (∞, r)-category is an infinity category, with the additional requirement that all j-morphisms, $j > r$, be invertible. In this subsection, we actually confine ourselves to $(\infty, 1)$*-categories, which we simply call* ∞*-categories.*

5.4.2.1 First examples

Of course,

Example 5.4.1. ∞*-categories should include ordinary categories.*

There is another natural example of infinity category. When considering all the paths in $T \in \mathtt{Top}$, up to homotopy (for fixed initial and final points), we obtain the *fundamental groupoid* $\Pi_1(T)$ of T. Remember that the usual 'half-time' composition of paths is not associative, whereas the induced composition of homotopy classes is. Hence, $\Pi_1(T)$, with the points of T as objects and the homotopy classes of paths as morphisms, is (really) a category in which all morphisms are invertible. To encode more information about T, we can use a 2-category, the fundamental 2-groupoid $\Pi_2(T)$, whose 0-morphisms (resp., 1-morphisms, 2-morphisms) are the points of T (resp., the paths between points, the homotopy classes of homotopies between paths), the composition of 1-morphisms being associative only up to a 2-isomorphism. More generally, we define the *fundamental k-groupoid* $\Pi_k(T)$, in which associativity of order $j \leq k-1$ holds up to a $(j+1)$-isomorphism. Of course, if we increase k, we grasp more and more information about T. The *homotopy principle* says that the weak fundamental infinity groupoid $\Pi_\infty(T)$ recognizes T, or, more precisely, that $(\infty, 0)$*-categories are the same as topological spaces.* Hence,

Example 5.4.2. ∞-categories, i.e. $(\infty, 1)$-categories, should contain topological spaces.

5.4.2.2 Kan complexes, quasi-categories, nerves of groupoids and of categories

Let us recall that the *nerve functor* $N : \mathtt{Cat} \to \mathtt{SSet}$, provides a fully faithful embedding of the category $\mathtt{Cat}$ of all (small) categories into $\mathtt{SSet}$ and remembers not only the objects and morphisms, but also the compositions. It associates to any $\mathtt{C} \in \mathtt{Cat}$ the simplicial set

$$(N\mathtt{C})_n = \{C_0 \to C_1 \to \ldots \to C_n\}\ ,$$

where the sequence in the RHS is a sequence of composable $\mathtt{C}$-morphisms between objects $C_i \in \mathtt{C}$; the face (resp., the degeneracy) maps are the compositions and insertions of identities. Let us also recall that the *r-horn* $\Lambda^r[n]$, $0 \le r \le n$, of $\Delta[n]$ is 'the part of the boundary of $\Delta[n]$ that is obtained by suppressing the interior of the $(n-1)$-face opposite to r'. More precisely, the r-horn $\Lambda^r[n]$ is the simplicial set, whose nondegenerate k-simplices are the injective order-respecting maps $[k] \to [n]$, except the identity and the map $\delta^r : [n-1] \rightarrowtail [n]$ whose image does not contain r.

We now detail four different situations based on the properties 'Any (inner) horn admits a (unique) filler'.

Definition 5.4.3. A simplicial set $S \in \mathtt{SSet}$ is *fibrant* and called a *Kan complex*, if the map $S \to \star$, where $\star$ denotes the terminal object, is a Kan fibration, i.e. has the right lifting property with respect to all canonical inclusions $\Lambda^r[n] \subset \Delta[n]$, $0 \le r \le n$, $n > 0$. In other words, S is a Kan complex, if any horn $\Lambda^r[n] \to S$ can be extended to an n-simplex $\Delta[n] \to S$, i.e. if any horn in S admits a filler.

The following result is well-known and explains that a simplicial set is a nerve under a quite similar extension condition.

Proposition 5.4.4. *A simplicial set S is the nerve $S \simeq N\mathtt{C}$ of some category $\mathtt{C}$, if and only if* any inner horn $\Lambda^r[n] \to S$, $0 < r < n$, has a unique filler $\Delta[n] \to S$.

Indeed, it is quite obvious that for $S = N\mathtt{C} \in \mathtt{SSet}$, an inner horn $\Lambda^1[2] \to N\mathtt{C}$, i.e. two $\mathtt{C}$-morphisms $f : C_0 \to C_1$ and $g : C_1 \to C_2$, has a unique filler $\Delta[2] \to N\mathtt{C}$, given by the edge $h = g \circ f : C_0 \to C_2$ and the 'homotopy' $\mathrm{id} : h \Rightarrow g \circ f$ (1).
As for Kan complexes $S \in \mathtt{SSet}$, the filler property for an outer horn $\Lambda^0[2] \to S$ (resp., $\Lambda^2[2] \to S$) implies for instance that a horn $f : s_0 \to s_1$, $\mathrm{id} : s_0 \to s_2 = s_0$ (resp., $\mathrm{id} : s_0' \to s_2' = s_0'$, $g : s_1' \to s_2'$) has a filler, so that any map has a 'left (resp., right) inverse' (2).

It is clear that simplicial sets $S_0, S_1, S_2, \dots$ are candidates for ∞-categories. In view of the last remark (2), Kan complexes model ∞-groupoids. Hence, fillers for outer horns should be omitted in the definition of ∞-categories. On the other hand, ∞-categories do contain homotopies $\eta : h \Rightarrow g \circ f$, so that, due to (1), uniqueness of fillers is to be omitted as well. Hence, the

Definition 5.4.5. A simplicial set $S \in \mathtt{SSet}$ is an *∞-category* if and only if any inner horn $\Lambda^r[n] \to S$, $0 < r < n$, admits a filler $\Delta[n] \to S$.

We now also understand the

Proposition 5.4.6. *A simplicial set S is the nerve $S \simeq N\mathtt{G}$ of some groupoid $\mathtt{G}$, if and only if* any horn $\Lambda^r[n] \to S$, $0 \le r \le n$, $n > 0$, has a unique filler $\Delta[n] \to S$.

Of course, Kan complexes, i.e. ∞-groupoids, $(\infty, 0)$-categories, or, still, topological spaces, are ∞-categories. Moreover, nerves of categories are ∞-categories. Hence, the requirement that topological spaces and ordinary categories should be ∞-categories are satisfied. Note further that what we just defined is a model for ∞-categories called *quasi-categories* or *weak Kan complexes*.

5.4.2.3 Link with the intuitive picture of an infinity category

In the following, we explain that the preceding model of an ∞-category actually corresponds to the intuitive picture of an $(\infty, 1)$-category, i.e. that in an ∞-category all types of morphisms do exist, that all j-morphisms, $j > 1$, are invertible, and that composition of morphisms is defined and is associative only up to homotopy. This will be illustrated by showing that any ∞-category has a homotopy category, which is an ordinary category.

We denote simplicial sets by $S, S', \ldots$, categories by $\mathtt{C}, \mathtt{D}, \ldots$, and ∞-categories by $\mathtt{S}, \mathtt{S}', \ldots$

Let $\mathtt{S}$ be an ∞-category. Its *0-morphisms* are the elements of $\mathtt{S}_0$ and its *1-morphisms* are the elements of $\mathtt{S}_1$. The *source* and *target* maps σ, τ are defined, for any 1-morphism $f \in \mathtt{S}_1$, by $\sigma f = d_1 f \in \mathtt{S}_0$, $\tau f = d_0 f \in \mathtt{S}_0$, and the *identity map* is defined, for any 0-morphism $s \in \mathtt{S}_0$, by $\mathrm{id}_s = s_0 s \in \mathtt{S}_1$, with self-explaining notation. In the following, we denote a 1-morphism f with source s and target s' by $f : s \to s'$. In view of the simplicial relations, we have $\sigma \mathrm{id}_s = d_1 s_0 s = s$ and $\tau \mathrm{id}_s = d_0 s_0 s = s$, so that $\mathrm{id}_s : s \to s$.

Consider now two morphisms $f : s \to s'$ and $g : s' \to s''$. They define an inner horn $\Lambda^1[2] \to \mathtt{S}$, which, as $\mathtt{S}$ is an ∞-category, admits a filler $\phi : \Delta[2] \to \mathtt{S}$, or $\phi \in \mathtt{S}_2$. The face $d_1 \phi \in \mathtt{S}_1$ is of course a candidate for the composite $g \circ f$.

Remark 5.4.7. Since the face $h := d_1 \phi$ of any filler ϕ is a (candidate for the) composite $g \circ f$, composites of morphisms are in ∞-categories not uniquely defined. We will show that they are determined only up to 'homotopy'.

Definition 5.4.8. Let $\mathtt{S}$ be an ∞-category and let $f, g : s \to s'$ be two morphisms. A *2-morphism* or *homotopy* $\phi : f \Rightarrow g$ between f and g is an element $\phi \in \mathtt{S}_2$ such that $d_0 \phi = g, d_1 \phi = f, d_2 \phi = \mathrm{id}_s$.

Indeed, if there exists such a 2-simplex ϕ, there are two candidates for the composite $g \circ \mathrm{id}_s$, namely f and, of course, g. If we wish now that all the candidates be homotopic, the existence of ϕ must entail that f and g are homotopic – which is the case in view of Definition 5.4.8. If f is homotopic to g, we write $f \simeq g$.

Proposition 5.4.9. *The homotopy relation $\simeq$ is an equivalence in $\mathtt{S}_1$.*

Proof. Let $f : s \to s'$ be a morphism and consider $\mathrm{id}_f := s_0 f \in \mathtt{S}_2$. It follows from the simplicial relations that $d_0 \mathrm{id}_f = f, d_1 \mathrm{id}_f = f, d_2 \mathrm{id}_f = s_0 s = \mathrm{id}_s$, so that id_f is a homotopy between f and f. To prove that $\simeq$ is symmetric, let $f, g : s \to s'$ and assume that ϕ is a homotopy from f to g. We then have an inner horn $\psi : \Lambda^2[3] \to \mathtt{S}$ such that $d_0 \psi = \phi$, $d_1 \psi = \mathrm{id}_g$, and $d_3 \psi = \mathrm{id}_{\mathrm{id}_s} =: \mathrm{id}_s^2$. The face $d_2 \Psi$ of a filler $\Psi : \Delta[3] \to \mathtt{S}$ is a homotopy from g to f. Transitivity can be obtained similarly. $\square$

Definition 5.4.10. The *homotopy category* $\mathtt{Ho}(\mathtt{S})$ of an ∞-category $\mathtt{S}$ is the (ordinary) category with objects the objects $s \in \mathtt{S}_0$, with morphisms the homotopy classes $[f]$ of morphisms $f \in \mathtt{S}_1$, with composition $[g] \circ [f] = [g \circ f]$, where $g \circ f$ is any candidate for the composite in $\mathtt{S}$, and with identities $\mathrm{Id}_s = [\mathrm{id}_s]$.

To check that this definition makes sense, we must in particular show that all composites $g \circ f$, see Remark 5.4.7, are homotopic. Let thus $\phi_1, \phi_2 \in \mathtt{S}_2$ be two 2-simplices such that $(d_0 \phi_1, d_1 \phi_1, d_2 \phi_1) = (g, h_1, f)$ and $(d_0 \phi_2, d_1 \phi_2, d_2 \phi_2) = (g, h_2, f)$, so that h_1 and h_2 are two candidates. Consider now for instance the inner horn $\psi : \Lambda^2[3] \to \mathtt{S}$ given by $\psi = (\phi_1, \phi_2, \bullet, \mathrm{id}_f)$.

The face $d_2\Psi$ of a filler $\Psi : \Delta[3] \to \mathtt{S}$ is then a homotopy from h_2 to h_1. To prove that the composition of morphisms in $\mathtt{Ho}(\mathtt{S})$ is associative, one shows that candidates for $h \circ (g \circ f)$ and for $(h \circ g) \circ f$ are homotopic (we will prove neither this fact, nor the additional requirements for $\mathtt{Ho}(\mathtt{S})$ to be a category).

Remark 5.4.11. It follows that in an ∞-category composition of morphisms is defined and associative only up to homotopy.

We now comment on higher morphisms in ∞-categories, on their composites, as well as on invertibility of j-morphisms, $j > 1$.

Definition 5.4.12. Let $\phi_1 : f \Rightarrow g$ and $\phi_2 : f \Rightarrow g$ be 2-morphisms between morphisms $f, g : s \to s'$. A *3-morphism* $\Phi : \phi_1 \Rrightarrow \phi_2$ is an element $\Phi \in \mathtt{S}_3$ such that $d_0\Phi = \mathrm{id}_g$, $d_1\Phi = \phi_2$, $d_2\Phi = \phi_1$, and $d_3\Phi = \mathrm{id}^2_s$.

Roughly, a 3-morphism is a 3-simplex with faces given by sources and targets, as well as by identities. Higher morphisms are defined similarly [Gro10].

As concerns composition and invertibility, let us come back to transitivity of the homotopy relation. There we are given 2-morphisms $\phi_1 : f \Rightarrow g$ and $\phi_2 : g \Rightarrow h$, and must consider the inner horn $\psi = (\phi_2, \bullet, \phi_1, \mathrm{id}^2_s)$. The face $d_1\Psi$ of a filler Ψ is a homotopy between f and h and is a candidate for the composite $\phi_2 \circ \phi_1$ of the 2-morphisms ϕ_1, ϕ_2. If we now look again at the proof of symmetry of the homotopy relation and denote the homotopy from g to f by ψ', we see that $\psi \circ \psi' \simeq \mathrm{id}_g$. We obtain similarly that $\psi' \circ \psi \simeq \mathrm{id}_f$, so that 2-morphisms are 'invertible'.

Remark 5.4.13. Eventually, all the requirements of the intuitive picture of an ∞-category are (really) encoded in the existence of fillers of inner horns.

5.4.3 Infinity groupoid of infinity morphisms between Leibniz infinity algebras

5.4.3.1 Quasi-category of homotopy Leibniz algebras Let $\Omega^\star_\bullet$ be the SDGCA introduced in Subsection 5.3.2.2. The 'Yoneda embedding' of $\Omega^\star_\bullet$ viewed as object of $\mathtt{SSet}$ and $\mathtt{DGCA}$, respectively, gives rise to an adjunction that is well-known in Rational Homotopy Theory:

$$\Omega^\star : \mathtt{SSet} \rightleftarrows \mathtt{DGCA}^{\mathrm{op}} : \mathrm{Spec}_\bullet \ .$$

The functor $\Omega^\star = \mathrm{Hom}_{\mathtt{SSet}}(-, \Omega^\star_\bullet) =: \mathrm{SSet}(-, \Omega^\star_\bullet)$ associates to any $S_\bullet \in \mathtt{SSet}$ its Sullivan DGCA $\Omega^\star(S_\bullet)$ of piecewise polynomial differential forms, whereas the functor $\mathrm{Spec}_\bullet = \mathrm{Hom}_{\mathtt{DGCA}}(-, \Omega^\star_\bullet)$ assigns to any $A \in \mathtt{DGCA}$ its simplicial spectrum $\mathrm{Spec}_\bullet(A)$.

Remember now that an ∞-homotopy between ∞-morphisms between two Leibniz infinity algebras V, W, is an element in $\mathrm{MC}_1(\bar{L}) = \mathrm{MC}(L \otimes \Omega^\star_1)$, where $L = L(V, W)$.

The latter set is well-known from integration of L_∞-algebras. Indeed, when looking for an integrating topological space or simplicial set of a positively graded L_∞-algebra L of finite type (degree-wise of finite dimension), it is natural to consider the simplicial spectrum of the corresponding quasi-free DGCA $\mathrm{Com}(s^{-1}L^*)$. The dual of Equation (5.19) yields

$$\mathrm{Spec}_\bullet(\mathrm{Com}(s^{-1}L^*)) = \mathrm{Hom}_{\mathtt{DGCA}}(\mathrm{Com}(s^{-1}L^*), \Omega^\star_\bullet) \simeq \mathrm{MC}(L \otimes \Omega^\star_\bullet) \ .$$

The integrating simplicial set of a nilpotent L_∞-algebra L is actually homotopy equivalent to $\mathrm{MC}_\bullet(\bar{L}) := \mathrm{MC}(L \otimes \Omega^\star_\bullet)$ [Get09]. It is clear that the structure maps of $\mathrm{MC}_\bullet(\bar{L}) \subset L \otimes \Omega^\star_\bullet$ are $\tilde{d}^n_i = \mathrm{id} \otimes d^n_i$ and $\tilde{s}^n_i = \mathrm{id} \otimes s^n_i$, where d^n_i and s^n_i were described in Subsection 5.3.2.2.

Higher homotopies (n-homotopies) are usually defined along the same lines as standard homotopies (1-homotopies), i.e., e.g., as arrows depending on parameters in $I^{\times n}$ (or Δ^n) instead of I (or Δ^1) [Lei04]. Hence,

Definition 5.4.14. *∞-n-homotopies* (∞-$(n+1)$-*morphisms*) between given Leibniz infinity algebras V, W are Maurer-Cartan elements in $\mathrm{MC}_n(\bar{L}) = \mathrm{MC}(L \otimes \Omega^\star_n)$, where $L = L(V, W)$ and $n \geq 0$.

Indeed, ∞-1-morphisms are just elements of $\mathrm{MC}(L)$, i.e. standard ∞-morphisms between V and W.

Note that if $\mathtt{S}$ is an ∞-category, the set of n-morphisms, with varying $n \geq 1$, between two fixed objects $s, s' \in \mathtt{S}_0$ can be shown to be a Kan complex [Gro10]. The simplicial set $\mathrm{MC}_\bullet(\bar{L})$, whose $(n-1)$-simplices are the ∞-n-morphisms between the considered Leibniz infinity algebras V, W, $n \geq 1$, is known to be a Kan complex ($(\infty, 0)$-category) as well [Get09].

Remark 5.4.15. We interpret this result by saying that Leibniz infinity algebras and their infinity higher morphisms form an ∞-category ($(\infty, 1)$-category). Further, as mentioned above and detailed below, composition of homotopies is encrypted in the Kan property.

Note that $\mathrm{MC}_\bullet(\bar{L})$ actually corresponds to the 'décalage', the 'down-shifting', of the simplicial set $\mathtt{S}$.

Let us also *emphasize* that Getzler's results are valid only for nilpotent L_∞-algebras, hence in principle not for L, which is only complete (an L_∞-algebra is *pronilpotent*, if it is complete with respect to its lower central series, i.e. the intersection of all its compatible filtrations, and it is *nilpotent*, if its lower central series eventually vanishes). However, for our remaining concern, namely the explanation of homotopies and their compositions in the 2-term Leibniz infinity algebra case, this difficulty is irrelevant. Indeed, when interpreting the involved series as formal ones and applying the thus obtained results to the 2-term case, where series become finite for degree reasons, we recover the results on homotopies and their compositions conjectured in [BC04]. An entirely rigorous approach to these issues is being examined in a separate paper: it is rather technical and requires applying Henriques' method or working over an arbitrary local Artinian algebra.

5.4.3.2 Kan property

Considering our next purposes, we now review and specify the proof of the Kan property of $\mathrm{MC}_\bullet(\bar{L})$ [Get09]. As announced above, to facilitate comparison, we adopt the conventions of the latter paper and apply the results mutatis mutandis and formally to our situation. In particular, we work in 5.4.3.2 with the cohomological version of Lie infinity algebras (k-ary bracket of degree $2-k$), together with Getzler's sign convention for the higher Jacobi conditions, see 5.3.2.1, and assume that $\mathbb{K} = \mathbb{R}$.

Let us first recall that the lower central filtration of $(L, \mathcal{L}_i)$ is given by $F^1 L = L$ and

$$F^i L = \sum_{i_1 + \cdots + i_k = i} \mathcal{L}_k(F^{i_1} L, \cdots, F^{i_k} L), \quad i > 1 .$$

In particular, $F^2L=\mathcal{L}_2(L,L)$, $F^3L=\mathcal{L}_2(L,\mathcal{L}_2(L,L))+\mathcal{L}_3(L,L,L)$, ..., so that F^kL is spanned by all the nested brackets containing k elements of L. Due to nilpotency, $F^iL=\{0\}$, for $i\gg$.

To simplify notation, let δ be the differential $\mathcal{L}_1$ of L, let d be the de Rham differential of $\Omega^\star_n=\Omega^\star(\Delta^n)$, and let $\bar\delta+\bar d$ be the differential $\bar{\mathcal{L}}_1=\delta\otimes\mathrm{id}+\mathrm{id}\otimes d$ of $L\otimes\Omega^\star(\Delta^n)$. Set now, for any $n\geq 0$ and any $0\leq i\leq n$,

$$\mathrm{mc}_n(\bar L):=\{(\bar\delta+\bar d)\beta:\beta\in(L\otimes\Omega^\star(\Delta^n))^0\}\text{ and }\mathrm{mc}^i_n(\bar L):=\{(\bar\delta+\bar d)\beta:\beta\in(L\otimes\Omega^\star(\Delta^n))^0,\ \bar\varepsilon^i_n\beta=0\}\ ,$$

where $\bar\varepsilon^i_n:=\mathrm{id}\otimes\varepsilon^i_n$ is the canonical extension of the evaluation map $\varepsilon^i_n:\Omega^\star(\Delta^n)\to\mathbb{K}$, see 5.3.2.2.

Remark 5.4.16. In the following, we use the extension symbol 'bar' only when needed for clarity.

- There exist fundamental bijections

$$B^i_n:\mathrm{MC}_n(\bar L)\stackrel{\sim}{\longrightarrow}\mathrm{MC}(L)\times\mathrm{mc}^i_n(\bar L)\subset\mathrm{MC}(L)\times\mathrm{mc}_n(\bar L)\ . \tag{5.23}$$

The proof uses the operators

$$h^i_n:\ \Omega^\star(\Delta^n)\to\Omega^{\star-1}(\Delta^n)$$

defined as follows. Let $\vec t=[t_0,\ldots,t_n]$ be the coordinates of Δ^n (with $\sum_i t_i=1$) and consider the maps $\phi^i_n:\ I\times\Delta^n\ni(u,\vec t)\mapsto u\vec t+(1-u)\vec e_i\in\Delta^n$. They allow to pull back a polynomial differential form on Δ^n to a polynomial differential form on $I\times\Delta^n$. The operators h^i_n are now given by

$$h^i_n\omega=\int_I\ (\phi^i_n)^*\omega\ .$$

They satisfy the relations

$$\{d,h^i_n\}=\mathrm{id}_n-\varepsilon^i_n,\qquad\{h^i_n,h^j_n\}=0,\qquad\varepsilon^i_nh^i_n=0\ , \tag{5.24}$$

where $\{-,-\}$ is the graded commutator (remember that ε^i_n vanishes in nonzero cohomological degree). The first relation is a higher dimensional analogue of

$$\{d,\int_0^t\}\omega=\{d,\int_0^t\}(f(u)+g(u)du)=d\int_0^t g(u)du+\int_0^t d_uf\,du=g(t)dt+f(t)-f(0)=\omega-\varepsilon^0_1\omega\ ,$$

where $\omega\in\Omega^\star(I)$.

The natural extensions of d, h^i_n, and ε^i_n to $L\otimes\Omega^\star(\Delta^n)$ verify the same relations, and, since we obviously have $\delta h^i_n=-h^i_n\delta$, the first relation holds in the extended setting also for d replaced by $\delta+d$.

Define now B^i_n by

$$B^i_n:\mathrm{MC}_n(\bar L)\ni\alpha\mapsto B^i_n\alpha:=(\varepsilon^i_n\alpha,(\delta+d)h^i_n\alpha)\in\mathrm{MC}(L)\times\mathrm{mc}^i_n(\bar L)\ . \tag{5.25}$$

Observe that $\alpha\in(L\otimes\Omega^\star(\Delta^n))^1$ reads $\alpha=\sum_{k=0}^n\alpha^k$, $\alpha^k\in L_{1-k}\otimes\Omega^k(\Delta^n)$, so that $\varepsilon^i_n\alpha=\varepsilon^i_n\alpha^0\in L_1$. Moreover, it follows from the definition of the extended L_∞-maps $\bar{\mathcal{L}}_i$ that

$$\sum_{i\geq1}\frac{1}{i!}\mathcal{L}_i(\varepsilon^i_n\alpha,\ldots,\varepsilon^i_n\alpha)=\varepsilon^i_n\sum_{i\geq1}\frac{1}{i!}\bar{\mathcal{L}}_i(\alpha,\ldots,\alpha)=0\ . \tag{5.26}$$

In view of the last equation (5.24), the second component of $B_n^i\alpha$ is clearly an element of $\mathrm{mc}_n^i(\bar{L})$.

The construction of the inverse map is based upon a method similar to the iterative approximation procedure that allows to prove the fundamental theorem of ODE-s. More precisely, consider the Cauchy problem $y'(t) = F(t, y(t))$, $y(0) = Y$, i.e. the integral equation

$$y(s) = Y + \int_0^s F(t, y(t))dt \; .$$

Choose now the 'Ansatz' $y_0(s) = Y$ and define inductively

$$y_k(s) = Y + \int_0^s F(t, y_{k-1}(t))dt \; ,$$

$k \geq 1$. It is well-known that the y_k converge to a function y, which is the unique solution and depends continuously on the initial value Y.

Note now that, if we are given $\mu \in \mathrm{MC}(L)$ and $\nu = (\delta + d)\beta \in \mathrm{mc}_n^i(\bar{L})$, a solution $\alpha \in \mathrm{MC}_n(\bar{L})$ – i.e. an element $\alpha \in (L \otimes \Omega^\star(\Delta^n))^1$ that satisfies

$$(\delta + d)\alpha + \sum_{i\geq 2} \frac{1}{i!}\bar{\mathcal{L}}_i(\alpha, \ldots, \alpha) =: (\delta + d)\alpha + \bar{R}(\alpha) = 0 \quad -$$

such that $\varepsilon_n^i\alpha = \mu$ and $(\delta + d)h_n^i\alpha = \nu$, also verifies the integral equation

$$\alpha = \mathrm{id}_n\alpha = \{\delta + d, h_n^i\}\alpha + \varepsilon_n^i\alpha = \mu + \nu + h_n^i(\delta + d)\alpha = \mu + \nu - h_n^i\bar{R}(\alpha) \; . \tag{5.27}$$

We thus choose the 'Ansatz' $\alpha_0 = \mu + \nu$ and set $\alpha_k = \alpha_0 - h_n^i\bar{R}(\alpha_{k-1})$, $k \geq 1$. It is easily seen that, in view of nilpotency, this iteration stabilizes, i.e. $\alpha_{k-1} = \alpha_k = \ldots =: \alpha$, for $k \gg$, or, still,

$$\alpha = \alpha_0 - h_n^i\bar{R}(\alpha) \; . \tag{5.28}$$

The limit α is actually a solution in $\mathrm{MC}_n(\bar{L})$. Indeed, remember first that the generalized curvature

$$\bar{\mathcal{F}}(\alpha) = (\delta + d)\alpha + \bar{R}(\alpha) = (\delta + d)\alpha + \sum_{i\geq 2} \frac{1}{i!}\bar{\mathcal{L}}_i(\alpha, \ldots, \alpha) \; ,$$

whose zeros are the MC elements, satisfies, just like the standard curvature, the Bianchi identity

$$(\delta + d)\bar{\mathcal{F}}(\alpha) + \sum_{k\geq 1} \frac{1}{k!}\bar{\mathcal{L}}_{k+1}(\alpha, \ldots, \alpha, \bar{\mathcal{F}}(\alpha)) = 0 \; . \tag{5.29}$$

It follows from (5.28) and (5.24) that

$$\bar{\mathcal{F}}(\alpha) = (\delta + d)(\alpha_0 - h_n^i\bar{R}(\alpha)) + \bar{R}(\alpha) = (\delta + d)\mu + h_n^i(\delta + d)\bar{R}(\alpha) + \varepsilon_n^i\bar{R}(\alpha) \; .$$

From Equation (5.26) we know that $\varepsilon_n^i\bar{R}(\alpha) = R(\varepsilon_n^i\alpha) = R(\mu)$, with self-explaining notation. As for $\varepsilon_n^i\alpha = \mu$, note that $\varepsilon_n^i\mu = \mu$ and that

$$\varepsilon_n^i\nu = \varepsilon_n^i(\delta{+}d)\beta = \varepsilon_n^i(\delta{+}d)\sum_{k=0}^n \beta^k, \;\; \beta^k \in L_{-k}{\otimes}\Omega^k(\Delta^n), \;\;\text{so that}\;\; \varepsilon_n^i\nu = \varepsilon_n^i\delta\beta^0 = \delta\varepsilon_n^i\beta^0 = \delta\varepsilon_n^i\beta = 0 \; .$$

Hence,

$$\begin{aligned}\bar{\mathcal{F}}(\alpha) &= (\delta+d)\mu + h^i_n(\delta+d)(\bar{\mathcal{F}}(\alpha)-(\delta+d)\alpha) + R(\mu) \\ &= \mathcal{F}(\mu) + h^i_n(\delta+d)\bar{\mathcal{F}}(\alpha) = -h^i_n\sum_{k\geq 1}\frac{1}{k!}\bar{\mathcal{L}}_{k+1}(\alpha,\ldots,\alpha,\bar{\mathcal{F}}(\alpha))\ ,\end{aligned}$$

in view of (5.29). Therefore, $\bar{\mathcal{F}}(\alpha)\in F^i\bar{L}$, for arbitrarily large i, and thus $\alpha\in\mathrm{MC}_n(\bar{L})$. This completes the construction of maps

$$\mathcal{B}^i_n : \mathrm{MC}(L)\times\mathrm{mc}^i_n(\bar{L})\to\mathrm{MC}_n(\bar{L})\ . \tag{5.30}$$

We already observed that $\varepsilon^i_n\mathcal{B}^i_n(\mu,\nu)=\varepsilon^i_n\alpha=\mu$. In fact, $B^i_n\mathcal{B}^i_n=\mathrm{id}$, so that B^i_n is surjective. Indeed, Equations (5.28) and (5.24) imply that

$$(\delta+d)h^i_n\alpha = -h^i_n(\delta+d)\alpha_0+\alpha_0-\varepsilon^i_n\alpha_0 = -h^i_n\delta\mu+\nu=\nu\ .$$

As for injectivity, if $B^i_n\alpha=B^i_n\alpha' =: (\mu,\nu)$, then both, α and α', satisfy Equation (5.27). It is now quite easily seen that nilpotency entails that $\alpha=\alpha'$.

- The bijections

$$B^i_n : \mathrm{MC}_n(\bar{L})\to\mathrm{MC}(L)\times\mathrm{mc}^i_n(\bar{L})$$

allow to prove the Kan property for $\mathrm{MC}_\bullet(\bar{L})$. The extension of a horn in $\mathrm{SSet}(\Lambda^i[n],\mathrm{MC}_\bullet(\bar{L}))$ will be performed as sketched by the following diagram:

$$\begin{array}{ccc} \mathrm{SSet}(\Lambda^i[n],\mathrm{MC}_\bullet(\bar{L})) & \dashrightarrow & \mathrm{MC}_n(\bar{L}) \\ \downarrow & & \uparrow \\ \mathrm{SSet}(\Lambda^i[n],\mathrm{MC}(L)\times\mathrm{mc}_\bullet(\bar{L})) & \longrightarrow & \mathrm{MC}(L)\times\mathrm{mc}^i_n(\bar{L}) \end{array} \tag{5.31}$$

Of course, the right arrow is nothing but $\mathcal{B}^i_n$.

$\star$ As for the left arrow, imagine, for simplicity, that $i=1$ and $n=2$, and let

$$\alpha\in\mathrm{SSet}(\Lambda^1[2],\mathrm{MC}_\bullet(\bar{L}))\ .$$

The restrictions $\alpha|_{01}$ and $\alpha|_{12}$ to the 1-faces 01, 12 (compositions of the natural injections with α) are elements in $\mathrm{MC}_1(\bar{L})$, so that the map B^1_1 sends $\alpha|_{01}$ to (μ,ν) in $\mathrm{MC}(L)\times\mathrm{mc}_1(\bar{L})$ (and similarly $B^0_1(\alpha|_{12})=(\mu',\nu')\in\mathrm{MC}(L)\times\mathrm{mc}_1(\bar{L})$). Of course, $\mu=\varepsilon^1_1(\alpha|_{01})=\varepsilon^0_1(\alpha|_{12})=\mu'$. Since $\nu=(\delta+d)\beta$ and $\beta(1)=\varepsilon^1_1\beta=0$, we find $\nu(1)=\varepsilon^1_1\nu=0$ (and similarly $\nu'=(\delta+d)\beta'$ and $\beta'(1)=\nu'(1)=0$). Thus,

$$(\mu;\nu,\nu')\in\mathrm{SSet}(\Lambda^1[2],\mathrm{MC}(L)\times\mathrm{mc}_\bullet(\bar{L}))\ ,$$

which explains the left arrow.

$\star$ For the bottom arrow, let again $i=1, n=2$. Since μ is constant, it can be extended to the whole simplex. To extend (ν,ν'), it actually suffices to extend (β,β'). Indeed, restriction obviously commutes with δ. As for commutation with d, remember that $\Omega^\star:\Delta^{\mathrm{op}}\to\mathtt{DGCA}$ and that the $\texttt{DGCA}$-map $d_2^2 = \Omega^\star(\delta_2^2)$ sets the component t_2 to 0. Hence, d_2^2 coincides with restriction to 01 and commutes with d. Let now $\bar\beta$ be an extension of (β, β'). Since

$$(d\bar\beta)|_{01} = d_2^2 d\bar\beta = dd_2^2\bar\beta = d\beta$$

and similarly $(d\bar\beta)|_{12} = d\beta'$.

It now remains to explain that an extension $\bar\beta$ does always exist. Consider the slightly more general extension problem of three polynomial differential forms β_0, β_1, and β_2 defined on the 1-faces 12, 02, and 01 of the 2-simplex Δ^2, respectively (it is assumed that they coincide at the vertices). Let $\pi_2 : \Delta^2 \to 01$ be the projection defined, for any $\vec{t} = [t_0, t_1, t_2]$, as the intersection of the line $u\vec{t} + (1-u)\vec{e}_2$ with 01. This projection is of course ill-defined at $\vec{t} = \vec{e}_2$. In coordinates, we get

$$\pi_2 : [t_0, t_1, t_2] \mapsto [t_0/(1-t_2), t_1/(1-t_2)] \ .$$

It follows that the pullback $\pi_2^*\beta_2$ is a rational differential form with denominator $(1-t_2)^N$, for some integer N. Hence,

$$\gamma_2 := (1-t_2)^N \pi_2^*\beta_2$$

is a polynomial differential form on Δ^2 that coincides with β_2 on 01. It now suffices to solve the same extension problem as before, but for the forms $\beta_0 - \gamma_2|_{12}$, $\beta_1 - \gamma_2|_{02}$, and 0. When iterating the procedure – due to Renshaw [Sul77] –, the problem reduces to the extension of $0, 0, 0$ (since the pullback preserves 0). This completes the description of the bottom arrow, as well as the proof of the Kan property of $\mathrm{MC}_\bullet(\bar L)$.

5.5 2-Category of 2-term Leibniz infinity algebras

Categorification replaces sets (resp., maps, equations) by categories (resp., functors, natural isomorphisms). In particular, rather than considering two maps as equal, one details a way of identifying them. Categorification is thus a sharpened viewpoint that turned out to provide deeper insight. This motivates the interest in e.g. categorified algebras (and in truncated infinity algebras – see below).

Categorified Lie algebras were introduced under the name of Lie 2-algebras in [BC04] and further studied in [Roy07], [SL10], and [KMP11]. The main result of [BC04] states that Lie 2-algebras and 2-term Lie infinity algebras form equivalent 2-categories. However, **infinity homotopies of 2-term Lie infinity algebras** (resp., **compositions of such homotopies**) are not explained, but appear as some God-given natural transformations read through this EQUIVALENCE (resp., compositions are addressed only in [SS07] and performed in the ALGEBRAIC OR COALGEBRAIC SETTINGS).

This circumstance is not satisfactory, and **the attempt to improve our understanding of infinity homotopies and their compositions is one of the main concerns of the present paper**. Indeed, in [KMP11] (resp., [BP12]), the authors show that the EQUIVALENCE between n-term Lie infinity algebras and Lie n-algebras is, for $n > 2$, not as straightforward as expected – which is essentially due to the largely ignored fact that the category $\texttt{Vect}\ n\texttt{-Cat}$ of linear n-categories is symmetric monoidal, but that the corresponding map $\boxtimes : L \times L' \to L \boxtimes L'$ is not an n-functor (resp., that the understanding of a concept in the ALGEBRAIC FRAMEWORK is far from implying its comprehension in the infinity context – a reality that is corroborated e.g. by the comparison of concordances and infinity homotopies).

In this section, we obtain **explicit formulae for infinity homotopies and their compositions**, applying the KAN PROPERTY of $\mathrm{MC}_\bullet(\bar{L})$ to the 2-term case, thus staying inside the INFINITY SETTING.

5.5.1 Category of 2-term Leibniz infinity algebras

For the sake of completeness, we first describe 2-term Leibniz infinity algebras and their morphisms. Propositions 5.5.1 and 5.5.2 are specializations to the 2-term case of Definitions 5.2.6 and 5.2.8; see also [SL10]. The informed reader may skip the present subsection.

Proposition 5.5.1. *A* 2-term Leibniz infinity algebra *is a graded vector space $V = V_0 \oplus V_1$ concentrated in degrees 0 and 1, together with a linear, a bilinear, and a trilinear map l_1, l_2, and l_3 on V, of degree $|l_1| = -1$, $|l_2| = 0$, and $|l_3| = 1$, which verify, for any $w, x, y, z \in V_0$ and $h, k \in V_1$,*

(a) $l_1 l_2(x, h) = l_2(x, l_1 h)$,
$l_1 l_2(h, x) = l_2(l_1 h, x)$,

(b) $l_2(l_1 h, k) = l_2(h, l_1 k)$,

(c) $l_1 l_3(x, y, z) = l_2(x, l_2(y, z)) - l_2(y, l_2(x, z)) - l_2(l_2(x, y), z)$,

(d) $l_3(x, y, l_1 h) = l_2(x, l_2(y, h)) - l_2(y, l_2(x, h)) - l_2(l_2(x, y), h)$,
$l_3(x, l_1 h, y) = l_2(x, l_2(h, y)) - l_2(h, l_2(x, y)) - l_2(l_2(x, h), y)$,
$l_3(l_1 h, x, y) = l_2(h, l_2(x, y)) - l_2(x, l_2(h, y)) - l_2(l_2(h, x), y)$,

(e) $l_2(l_3(w, x, y), z) + l_2(w, l_3(x, y, z)) - l_2(x, l_3(w, y, z)) + l_2(y, l_3(w, x, z)) - l_3(l_2(w, x), y, z)$
$+ l_3(w, l_2(x, y), z) - l_3(x, l_2(w, y), z) - l_3(w, x, l_2(y, z)) + l_3(w, y, l_2(x, z)) - l_3(x, y, l_2(w, z)) = 0$.

Proposition 5.5.2. *An* infinity morphism between 2-term Leibniz infinity algebras (V, l_1, l_2, l_3) *and (W, m_1, m_2, m_3) is made up by a linear and a bilinear map f_1, f_2 from V to W, of degree $|f_1| = 0, |f_2| = 1$, which verify, for any $x, y, z \in V_0$ and $h \in V_1$,*

(a) $m_1 f_1 h = f_1 l_1 h$,

(b) $m_2(f_1 x, f_1 y) + m_1 f_2(x, y) = f_1 l_2(x, y)$,

(c) $m_2(f_1 x, f_1 h) = f_1 l_2(x, h) - f_2(x, l_1 h)$,
$m_2(f_1 h, f_1 x) = f_1 l_2(h, x) - f_2(l_1 h, x)$,

(d) $m_3(f_1 x, f_1 y, f_1 z) - m_2(f_2(x, y), f_1 z) + m_2(f_1 x, f_2(y, z)) - m_2(f_1 y, f_2(x, z)) =$
$f_1 l_3(x, y, z) + f_2(l_2(x, y), z) - f_2(x, l_2(y, z)) + f_2(y, l_2(x, z))$.

Corollary 5.5.3. *The category $\mathtt{2Lei}_\infty$ of 2-term Leibniz infinity algebras and infinity morphisms is a full subcategory of the category $\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$ of Leibniz infinity algebras and infinity morphisms.*

5.5.2 From the Kan property to 2-term infinity homotopies and their compositions

Definition 5.5.4. A *2-term infinity homotopy* between infinity morphisms $f = (f_1, f_2)$ and $g = (g_1, g_2)$, which act themselves between 2-term Leibniz infinity algebras (V, l_1, l_2, l_3) and (W, m_1, m_2, m_3), is a linear map θ_1 from V to W, of degree $|\theta_1| = 1$, which verifies, for any $x, y \in V_0$ and $h \in V_1$,

(a) $g_1 x - f_1 x = m_1 \theta_1 x$,

(b) $g_1 h - f_1 h = \theta_1 l_1 h$,

(c) $g_2(x, y) - f_2(x, y) = \theta_1 l_2(x, y) - m_2(f_1 x, \theta_1 y) - m_2(\theta_1 x, g_1 y)$.

The characterizing relations (a) - (c) of infinity Leibniz homotopies are the correct counterpart of the defining relations of infinity Lie homotopies [BC04]. However, rather than choosing the preceding relations as a mere definition, we deduce them here from the Kan property of $\mathrm{MC}_\bullet(\bar{L})$. More precisely,

Theorem 5.5.5. *There exist surjective maps S_1^i, $i \in \{0, 1\}$, from the class $\mathcal{I}$ of ∞-homotopies for 2-term Leibniz infinity algebras to the class $\mathcal{T}$ of 2-term ∞-homotopies for 2-term Leibniz infinity algebras.*

Remark 5.5.6. The maps S_1^i preserve the source and the target, i.e. they are surjections from the class $\mathcal{I}(f, g)$ of ∞-homotopies from f to g, to the class $\mathcal{T}(f, g)$ of 2-term ∞-homotopies from f to g. In the sequel, we refer to a preimage by S_1^i of an element $\theta_1 \in \mathcal{T}$ as a *lift* of θ_1 by S_1^i .

Proof. Henceforth, we use again the homological version of infinity algebras (k-ary bracket of degree $k - 2$), as well as the Lada-Stasheff sign convention for the higher Jacobi conditions and the MC equation.

Due to the choice of the homological variant of homotopy algebras, $\delta = \mathcal{L}_1$ has degree -1. For consistency, differential forms are then viewed as negatively graded; hence, $d : \Omega^{-k}(\Delta^n) \to \Omega^{-k-1}(\Delta^n)$, $k \in \{0, \dots, n\}$, and $\bar{\mathcal{L}}_1 = \delta \otimes \mathrm{id} + \mathrm{id} \otimes d$ has degree -1 as well. Similarly, the degree of the operator h_n^i is now $|h_n^i| = 1$. It is moreover easily checked that L cannot contain multilinear maps of nonnegative degree, i.e. that $L = \oplus_{k \geq 0} L_{-k}$. It follows that an element $\bar{\alpha} \in (L \otimes \Omega^\star(\Delta^n))^{-k}$, $k \geq 0$, reads

$$\bar{\alpha} = \sum \alpha_{-k} \otimes \omega^0 + \sum \alpha_{-k+1} \otimes \omega^{-1} + \dots \ ,$$

where the RHS is a finite sum. For instance, if $n = 2$, an element $\bar{\alpha}$ of degree -1 can be decomposed as

$$\bar{\alpha} = \alpha(s, t) \otimes 1 + \beta(s, t) \otimes ds + \beta'(s, t) \otimes dt \ ,$$

where (s, t) are coordinates of Δ^2 and where $\alpha(s, t) \in L_{-1}[s, t]$ and $\beta(s, t), \beta'(s, t) \in L_0[s, t]$ are polynomial functions in s, t with coefficients in L_{-1} and L_0, respectively.

In the sequel, we evaluate the L_∞-structure maps $\bar{\mathcal{L}}_i$ of $L \otimes \Omega^\star(\Delta^n)$ mainly on elements of degree -1 and 0, hence we compute the structure maps $\mathcal{L}_i$ of $L = \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(sV), W)$ on elements α

and β of degree -1 and 0, respectively. Let

$$\alpha = \sum_{p\geq 1} \alpha^p \in L \ , \quad |\alpha| = -1 \ ,$$
$$\beta = \sum_{p\geq 1} \beta^p \in L \ , \quad |\beta| = 0 \ ,$$

where $\alpha^p, \beta^p : \ (sV)^{\otimes p} \to W$. The point is that the concentration of V, W in degrees $0, 1$ entails that almost all components α^p, β^p vanish and that all series converge (which explains why the formal application of Getzler's method to the present situation leads to the correct counterpart of the findings of [BC04]). Indeed, the only nonzero components of α, β are

$$\begin{aligned} \alpha^1 : &\quad sV_0 \to W_0, \quad sV_1 \to W_1 \ , \\ \alpha^2 : &\quad (sV_0)^{\otimes 2} \to W_1 \ , \\ \beta^1 : &\quad sV_0 \to W_1 \ . \end{aligned} \tag{5.32}$$

Similarly, the nonzero components of the nonzero evaluations of the maps $\mathcal{L}_i$ on α-s and β-s are

$$\begin{aligned} \mathscr{L}_1(\alpha) : &\quad sV_1 \to W_0 \ , \quad (sV_0)^{\otimes 2} \to W_0 \ , \quad sV_0 \otimes sV_1 \to W_1 \ , \quad (sV_0)^{\otimes 3} \to W_1 \ , \\ \mathscr{L}_1(\beta) : &\quad sV_0 \to W_0 \ , \quad sV_1 \to W_1 \ , \quad (sV_0)^{\otimes 2} \to W_1 \ , \\ \mathscr{L}_2(\alpha_1, \alpha_2) : &\quad (sV_0)^{\otimes 2} \to W_0 \ , \quad sV_0 \otimes sV_1 \to W_1 \ , \quad (sV_0)^{\otimes 3} \to W_1 \ , \\ \mathscr{L}_2(\alpha, \beta) : &\quad (sV_0)^{\otimes 2} \to W_1 \ , \\ \mathscr{L}_3(\alpha_1, \alpha_2, \alpha_3) : &\quad (sV_0)^{\otimes 3} \to W_1 \ , \end{aligned} \tag{5.33}$$

see Proposition 5.3.8.

We are now prepared to concretize the iterative construction of $\mathcal{B}^i_n(\mu, \nu) \in \mathrm{MC}_n(\bar{L})$ from $\mu \in \mathrm{MC}(L)$ and $\nu = (\delta + d)\beta$, $\beta \in (L \otimes \Omega^\star(\Delta^n))^0$, $\varepsilon^i_n \beta = 0$ (the explicit forms of $\mathcal{B}^i_n(\mu, \nu)$ for $n = 1$ and $n = 2$ will be the main ingredients of the proofs of Theorems 5.5.5 and 5.5.7).

- Let $\alpha \in \mathcal{I}(f, g)$, i.e. let

$$\alpha \in \mathrm{MC}_1(\bar{L}) \xrightarrow{\sim} (\mu, (\delta + d)\beta) \in \mathrm{MC}(L) \times \mathrm{mc}^0_1(\bar{L}) \ ,$$

such that $\varepsilon^0_1 \alpha = f$ and $\varepsilon^1_1 \alpha = g$. To construct

$$\alpha = \mathcal{B}^0_1 B^0_1 \alpha = \mathcal{B}^0_1(\varepsilon^0_1 \alpha, (\delta + d) h^0_1 \alpha) =: \mathcal{B}^0_1(f, (\delta + d)\beta) =: \mathcal{B}^0_1(\mu, \nu) \ ,$$

we start from

$$\alpha_0 = \mu + (\delta + d)\beta \ .$$

The iteration unfolds as

$$\alpha_k = \alpha_0 - \sum_{j=2}^{\infty} \frac{1}{j!} h^0_1 \bar{\mathscr{L}}_j(\alpha_{k-1}, \cdots, \alpha_{k-1}) \ , \quad k \geq 1 \ .$$

Explicitly,

$$\begin{aligned}
\alpha_1 &= \mu + (\delta + d)\beta - \frac{1}{2} h_1^0 \bar{\mathscr{L}}_2(\alpha_0, \alpha_0) - \frac{1}{3!} h_1^0 \bar{\mathscr{L}}_3(\alpha_0, \alpha_0, \alpha_0) \\
&= \mu + (\delta + d)\beta - h_1^0 \bar{\mathscr{L}}_2(\mu + \delta\beta, d\beta) - \frac{1}{2} h_1^0 \bar{\mathscr{L}}_3(\mu + \delta\beta, \mu + \delta\beta, d\beta) \\
&= \mu + (\delta + d)\beta - h_1^0 \bar{\mathscr{L}}_2(\mu + \delta\beta, d\beta)\ .
\end{aligned}$$

Observe that $\mu + \delta\beta \in L_{-1}[t]$ and $d\beta \in L_0[t] \otimes dt$, that differential forms are concentrated in degrees 0 and -1, that h_1^0 annihilates 0-forms, and that the term in $\bar{\mathcal{L}}_3$ contains a factor of the type $\mathcal{L}_3(\alpha_1, \alpha_2, \beta)$ (notation of (5.33)), whose components vanish – see above. Analogously,

$$\begin{aligned}
\alpha_2 &= \mu + (\delta + d)\beta - h_1^0 \bar{\mathscr{L}}_2(\mu + \delta\beta - h_1^0 \bar{\mathscr{L}}_2(\mu + \delta\beta, d\beta), d\beta) \\
&\quad - \frac{1}{2} h_1^0 \bar{\mathscr{L}}_3(\mu + \delta\beta - h_1^0 \bar{\mathscr{L}}_2(\mu + \delta\beta, d\beta), \mu + \delta\beta - h_1^0 \bar{\mathscr{L}}_2(\mu + \delta\beta, d\beta), d\beta) \\
&= \mu + (\delta + d)\beta - h_1^0 \bar{\mathscr{L}}_2(\mu + \delta\beta, d\beta)\ .
\end{aligned}$$

Indeed, the term $h_1^0 \bar{\mathcal{L}}_2(h_1^0 \bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta), d\beta)$ contains a factor of the type $\mathcal{L}_2(\mathcal{L}_2(\alpha, \beta_1), \beta_2)$ (notation of (5.33)), and the only nonvanishing component of this factor, as well as of its first internal map $\mathcal{L}_2(\alpha, \beta_1)$, is the component $(sV_0)^{\otimes 2} \to W_1$ – which entails, in view of Proposition 5.3.8, that the considered term vanishes. Hence, the iteration stabilizes already at its second stage and

$$\alpha = \mathcal{B}_1^0(\mu, \nu) = \mu + (\delta + d)\beta - h_1^0 \bar{\mathscr{L}}_2(\mu + \delta\beta, d\beta) \in \mathrm{MC}_1(\bar{L})\ . \tag{5.34}$$

Remark first that the integral h_1^0 can be evaluated since $\bar{\mathscr{L}}_2(\mu + \delta\beta, d\beta)$ is a total derivative. Indeed, when setting $\beta = \beta_0 \otimes P$ (sum understood), $\beta_0 \in L_0$ and $P \in \Omega^0(\Delta^1)$, we see that

$$\bar{\mathcal{L}}_2(\mu, d\beta) = \mathcal{L}_2(\mu, \beta_0) \otimes dP = -d\bar{\mathcal{L}}_2(\mu, \beta)\ .$$

As for the term $\bar{\mathcal{L}}_2(\delta\beta, d\beta)$, we have

$$0 = (\delta + d)\bar{\mathcal{L}}_2(\beta, d\beta) = \bar{\mathscr{L}}_2(\delta\beta, d\beta) + \bar{\mathscr{L}}_2(\beta, \delta d\beta)\ ,$$

since $\bar{\mathcal{L}}_1 = \delta + d$ is a graded derivation of $\bar{\mathcal{L}}_2$ and as $\bar{\mathcal{L}}_2(\beta, d\beta) = \bar{\mathcal{L}}_2(d\beta, d\beta) = 0$. It is now easily checked that

$$\bar{\mathscr{L}}_2(\delta\beta, d\beta) = -\frac{1}{2} d\bar{\mathscr{L}}_2(\delta\beta, \beta)\ .$$

Eventually,

$$\begin{aligned}
\alpha &= \mu + (\delta + d)\beta + h_1^0 d\bar{\mathscr{L}}_2(\mu, \beta) + \frac{1}{2} h_1^0 d\bar{\mathscr{L}}_2(\delta\beta, \beta) \\
&= \mu + (\delta + d)\beta + \bar{\mathscr{L}}_2(\mu, \beta) + \frac{1}{2} \bar{\mathscr{L}}_2(\delta\beta, \beta)\ .
\end{aligned}$$

Indeed, it suffices to observe that, for any $\ell_{-1} \otimes P \in L_{-1} \otimes \Omega^0(\Delta^1)$ which vanishes under the action of ε_1^0, we have

$$h_1^0 d(\ell_{-1} \otimes P) = -dh_1^0(\ell_{-1} \otimes P) + \ell_{-1} \otimes P - \varepsilon_1^0(\ell_{-1} \otimes P) = \ell_{-1} \otimes P\ .$$

We are now able to write the components of $g = \varepsilon_1^1 \alpha \in L_{-1}$ (see (5.32)) in terms of $f = \mu$ and β:

$$g^1 = \varepsilon_1^1(\mu + (\delta + d)\beta - \bar{\mathscr{L}}_2(\mu, \beta) - \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta))^1 = \varepsilon_1^1(f + \delta\beta)^1 = f^1 + \varepsilon_1^1(\delta\beta)^1 \ ,$$
$$g^2 = \varepsilon_1^1(\mu + (\delta + d)\beta - \bar{\mathscr{L}}_2(\mu, \beta) - \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta))^2 = \varepsilon_1^1(f + \delta\beta - \bar{\mathscr{L}}_2(f, \beta) - \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta))^2 \ ,$$
$$g^3 = 0 \ , \tag{5.35}$$

where we changed signs according to our sign conventions and remembered that the first component of a morphism of the type $\mathcal{L}_2(\alpha, \beta)$ (see (5.33)) vanishes.

To obtain a 2-term ∞-homotopy $\theta_1 \in \mathcal{T}(f, g)$, it now suffices to further develop the equations (5.35).
As

$$g_1 := g^1 s, f_1 := f^1 s \in \mathrm{Hom}^0_{\mathbb{K}}(V, W) \ ,$$

we evaluate the first equation on $x \in V_0$ and $h \in V_1$. Therefore, we compute $\varepsilon_1^1(\delta\beta)^1 s = \delta\beta(1)^1 s$ on x and h. Since

$$\delta\beta(1) = \mathcal{L}_1\beta(1) = m_1\beta(1) + \beta(1)D_V \ ,$$

where $D_V \in \mathrm{CoDer}^{-1}(\mathrm{Zin}^c(sV))$, we have $D_V : sV_1 \to sV_0 \ , sV_0 \otimes sV_0 \to sV_0 \ , \ldots$ Hence,

$$\delta\beta(1)^1 sx = m_1\, \beta(1)s\, x = m_1\theta_1 x \ , \tag{5.36}$$

where we defined the *homotopy parameter* θ_1 by

$$\theta_1 := \beta(1)s = \beta(1)s - \beta(0)s \ . \tag{5.37}$$

Similarly,

$$\delta\beta(1)^1 sh = \beta(1)D_V sh = \beta(1)s\, s^{-1}D_V s\, h = \theta_1 l_1 h \ . \tag{5.38}$$

The characterizing equations (a) and (b) follow.
Since

$$g_2 := g^2 s^2, f_2 := f^2 s^2 \in \mathrm{Hom}^1_{\mathbb{K}}(V \otimes V, W) \ ,$$

it suffices to evaluate the second equation on $x, y \in V_0$. When computing e.g. $\varepsilon_1^1 \bar{\mathcal{L}}_2(\delta\beta, \beta)^2 s^2(x, y)$, we get

$$\mathcal{L}_2(\delta\beta(1), \beta(1))(sx, sy) = m_2(\delta\beta(1)\, sx, \beta(1)\, sy) + m_2(\beta(1)\, sx, \delta\beta(1)\, sy) =$$
$$m_2(m_1\theta_1 x, \theta_1 y) + m_2(\theta_1 x, m_1\theta_1 y) = 2m_2(\theta_1 x, m_1\theta_1 y) \ , \tag{5.39}$$

in view of Equation (5.36) and Relation (b) of Proposition 5.5.1. Similarly,

$$\varepsilon_1^1 \bar{\mathscr{L}}_2(f, \beta)^2 s^2(x, y) = m_2(f_1 x, \theta_1 y) + m_2(\theta_1 x, f_1 y) \ . \tag{5.40}$$

Further, one easily finds

$$\varepsilon_1^1(\delta\beta)^2 s^2(x, y) = \theta_1 l_2(x, y) \ . \tag{5.41}$$

When collecting the results (5.39), (5.40), and (5.41), and taking into account Relation (a), we finally obtain the characterizing equation (c).

- Recall that in the preceding step we started from $\alpha \in \mathcal{I}(f,g)$, set $\mu = f$,

$$\beta = h_1^0 \alpha \ ,$$

$\nu = (\delta + d)\beta$, defined

$$\theta_1 = (\beta(1) - \beta(0))s \ ,$$

and deduced the characterizing relations $g = f + \mathcal{E}(f, \beta(1)s) = f + \mathcal{E}(f, \theta_1)$ of $\theta_1 \in \mathcal{T}(f,g)$ by computing

$$\alpha = \mathcal{B}_1^0(\mu, \nu) = \mu + (\delta + d)\beta + \bar{\mathscr{L}}_2(\mu, \beta) + \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta)$$

at 1. Let us mention that instead of defining the map $S_1^0 : \mathcal{I} \ni \alpha \mapsto \theta_1 \in \mathcal{T}$, we can consider the similarly defined map S_1^1.

To prove surjectivity of S_1^i, let $\theta_1 \in \mathcal{T}(f,g)$ and set $\beta(i) = 0$, $i \in \{0,1\}$, and $\beta(1-i) = (-1)^i \theta_1 s^{-1}$. Note that by construction $\theta_1 = (\beta(1) - \beta(0))s$. Use now Renshaw's method [Sul77] to extend $\beta(0)$ and $\beta(1)$ to some $\beta \in L_0 \otimes \Omega^0(\Delta^1)$, set

$$\mu = (1-i)f + ig \quad \text{and} \quad \nu = (\delta + d)\beta \ ,$$

and construct

$$\alpha = \mathcal{B}_1^i(\mu, \nu) \in \mathrm{MC}_1(\bar{L}) \ . \tag{5.42}$$

If $i = 0$, then

$$\alpha(0) = \varepsilon_1^0 \alpha = \mu = f \quad \text{and} \quad \alpha(1) = (\mathcal{B}_1^0(\mu, \nu))(1) = f + \mathcal{E}(f, \theta_1) = g \ ,$$

in view of the characterizing relations (a)-(c) of θ_1. If $i = 1$, one has also $\alpha(1) = g$ and $\alpha(0) = g + \mathcal{E}(g, -\theta_1) = f$, but to obtain the latter result, the characterizing equations (a)-(c), as well as Equation (b) of Proposition 5.5.1 are needed. To determine the image of $\alpha \in \mathcal{I}(f,g)$ by S_1^i, one first computes $h_1^i \alpha$, which, since h_1^i sends 0-forms to 0, is equal to

$$h_1^i(\delta + d)\beta = -(\delta + d)h_1^i\beta + \beta - \varepsilon_1^i\beta = \beta \ ,$$

then one gets

$$S_1^i \alpha = (\beta(1) - \beta(0))s = \theta_1 \ ,$$

which completes the proof. □

Theorem 5.5.7 (Definition)**.** *If $\theta_1 : f \Rightarrow g$, $\tau_1 : g \Rightarrow h$ are 2-term ∞-homotopies between infinity morphisms $f, g, h : V \to W$, the vertical composite $\tau_1 \circ_1 \theta_1$ is given by $\tau_1 + \theta_1$.*

We will actually lift $\theta_1, \tau_1 \in \mathcal{T}$ to $\alpha', \alpha'' \in \mathrm{MC}_1(\bar{L})$ (which involves choices), then compose these lifts in the infinity groupoid $\mathrm{MC}_\bullet(\bar{L})$ (which is not a well-defined operation), and finally project the result back to $\mathcal{T}$ (despite all the intermediate choices, the final result will turn out to be well-defined).

Proof. Let now $n = 2$, take $\mu \in \mathrm{MC}(L)$ and $\nu = (\delta + d)\beta \in \mathrm{mc}_2^1(\bar{L})$, then construct $\alpha = \mathcal{B}_2^1(\mu, \nu)$. The computation is similar to that in the 1-dimensional case and gives the same result:

$$\alpha = \mu + (\delta + d)\beta + \bar{\mathscr{L}}_2(\mu, \beta) + \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta) \ . \tag{5.43}$$

To obtain $\tau_1 \circ_1 \theta_1$, proceed as in (5.42) and lift θ_1 (resp., τ_1) to

$$\alpha' := \mathcal{B}_1^1(g, (\delta+d)\beta') \in \mathcal{I}(f,g) \subset \mathrm{MC}_1(\bar{L}) \quad (\text{resp., } \alpha'' := \mathcal{B}_1^0(g, (\delta+d)\beta'') \in \mathcal{I}(g,h) \subset \mathrm{MC}_1(\bar{L}))\ ,$$

where

$$\beta'(0) = -\theta_1 s^{-1} \text{ and } \beta'(1) = 0 \quad (\text{resp., } \beta''(0) = 0 \text{ and } \beta''(1) = \tau_1 s^{-1})\ .$$

As mentioned above, we have by construction

$$\theta_1 = (\beta'(1) - \beta'(0))s \quad (\text{resp., } \tau_1 = (\beta''(1) - \beta''(0))s)\ . \tag{5.44}$$

If we view α' (resp., α'') as defined on the face 01 (resp., 12) of Δ^2, the equation $\varepsilon_1^1\alpha' = \varepsilon_1^0\alpha'' = g$ reads $\varepsilon_2^1\alpha' = \varepsilon_2^1\alpha'' = g =: \mu$. This means that

$$(\alpha', \alpha'') \in \mathrm{SSet}(\Lambda^1[2], \mathrm{MC}_\bullet(\bar{L}))\ .$$

We now follow the extension square (5.31). The left arrow leads to

$$(\mu; (\delta+d)\beta', (\delta+d)\beta'') \in \mathrm{SSet}(\Lambda^1[2], \mathrm{MC}(L) \times \mathrm{mc}_\bullet(\bar{L}))\ ,$$

the bottom arrow to

$$(\mu, (\delta+d)\beta) \in \mathrm{MC}(L) \times \mathrm{mc}_2^1(\bar{L})\ ,$$

where β is *any extension* of (β', β'') to Δ^2, and the right arrow provides $\alpha \in \mathrm{MC}_2(\bar{L})$ given by Equation (5.43). From Subsection 5.4.2.3, we know that all composites of α', α'' are ∞-2-homotopic and that a possible composite is obtained by restricting α to 02. This restriction $(-)|_{02}$ is given by the $\mathtt{DGCA}$-map d_1^2. Hence, we get

$$\alpha|_{02} = \mu + (\delta+d)\beta|_{02} + \bar{\mathscr{L}}_2(\mu, \beta|_{02}) + \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta|_{02}, \beta|_{02}) \in \mathcal{I}(f,h) \subset \mathrm{MC}_1(\bar{L})\ .$$

We now choose the projection $S_1^0\alpha|_{02} \in \mathcal{T}(f,h)$ of the composite-candidate of the chosen lifts of θ_1, τ_1, as composite $\tau_1 \circ_1 \theta_1$. Since

$$h_1^0\alpha|_{02} = -(\delta+d)h_1^0\beta|_{02} + \beta|_{02} - \beta(0) = \beta|_{02} - \beta(0)\ ,$$

we get

$$S_1^0\alpha|_{02} = (\beta|_{02}(2) - \beta(0) - \beta|_{02}(0) + \beta(0))s = (\beta(2) - \beta(0))s =$$
$$(\beta''(2) - \beta''(1))s + (\beta'(1) - \beta'(0))s = \tau_1 + \theta_1\ ,$$

in view of (5.44). Hence, by definition, the vertical composite of $\theta_1 \in \mathcal{T}(f,g)$ and $\tau_1 \in \mathcal{T}(g,h)$ is given by

$$\tau_1 \circ_1 \theta_1 = \tau_1 + \theta_1 \in \mathcal{T}(f,h)\ . \tag{5.45}$$

□

Remark 5.5.8. The composition of elements of $\mathcal{I} = \mathrm{MC}_1(\bar{L})$ in the infinity groupoid $\mathrm{MC}_\bullet(\bar{L})$, which is defined and associative only up to higher morphisms, projects to a well-defined and associative vertical composition in $\mathcal{T}$.

Just as for concordances, horizontal composition of ∞-homotopies is without problems. The horizontal composite of $\theta_1 \in \mathcal{T}(f,g)$ and $\tau_1 \in \mathcal{T}(f',g')$, where $f,g : V \to W$ and $f',g' : W \to X$ act between 2-term Leibniz infinity algebras, is defined by

$$\tau_1 \circ_0 \theta_1 = g_1' \theta_1 + \tau_1 f_1 = f_1' \theta_1 + \tau_1 g_1 \ . \tag{5.46}$$

The two definitions coincide, since θ_1, τ_1 are chain homotopies between the chain maps f,g and f',g', respectively, see Definition 5.5.4, Relations (a) and (b). The identity associated to a 2-term ∞-morphism is just the zero-map. As announced in [BC04] (in the Lie case and without information about composition), we have the

Proposition 5.5.9. *There is a strict 2-category* $\mathtt{2Lei}_\infty\text{-}\mathtt{Alg}$ *of 2-term Leibniz infinity algebras.*

5.6 2-Category of categorified Leibniz algebras

5.6.1 Category of Leibniz 2-algebras

Leibniz 2-algebras are categorified Leibniz structures on a categorified vector space. More precisely,

Definition 5.6.1. A *Leibniz 2-algebra* $(L, [-,-], \mathbf{J})$ is a linear category L equipped with

1. a *bracket* $[-,-]$, i.e. a bilinear functor $[-,-] : L \times L \to L$, and

2. a *Jacobiator* $\mathbf{J}$, i.e. a trilinear natural transformation

$$\mathbf{J}_{x,y,z} : [x,[y,z]] \to [[x,y],z] + [y,[x,z]], \quad x,y,z \in L_0,$$

which verify, for any $w, x, y, z \in L_0$, the *Jacobiator identity*

$$
\begin{array}{ccc}
 & [w,[x,[y,z]]] & \\
\swarrow {\scriptstyle [\mathbf{1}_w, \mathbf{J}_{x,y,z}]} & & {\scriptstyle \mathbf{1}} \searrow \\
[w,[[x,y],z]] + [w,[y,[x,z]]] & & [w,[x,[y,z]]] \\
\downarrow {\scriptstyle \mathbf{J}_{w,[x,y],z} + \mathbf{J}_{w,y,[x,z]}} & & \downarrow {\scriptstyle \mathbf{J}_{w,x,[y,z]}} \\
\begin{array}{c}[[w,[x,y]],z] + [[x,y],[w,z]] \\ +[[w,y],[x,z]] + [y,[w,[x,z]]]\end{array} & & [[w,x],[y,z]] + [x,[w,[y,z]]] \\
\downarrow {\scriptstyle \mathbf{1} + [\mathbf{1}_y, \mathbf{J}_{w,x,z}]} & & \downarrow {\scriptstyle \mathbf{1} + [\mathbf{1}_x, \mathbf{J}_{w,y,z}]} \\
\begin{array}{c}[[w,[x,y]],z] + [[x,y],[w,z]] \\ +[[w,y],[x,z]] + [y,[[w,x],z]] \\ +[y,[x,[w,z]]]\end{array} & & \begin{array}{c}[[w,x],[y,z]] + [x,[[w,y],z]] \\ +[x,[y,[w,z]]]\end{array} \\
{\scriptstyle [\mathbf{J}_{w,x,y}, \mathbf{1}_z]} \searrow & & \swarrow {\scriptstyle \mathbf{J}_{[w,x],y,z} + \mathbf{J}_{x,[w,y],z} + \mathbf{J}_{x,y,[w,z]}} \\
 & \begin{array}{c}[[[w,x],y],z] + [[x,[w,y]],z] \\ +[[x,y],[w,z]] + [[w,y],[x,z]] \\ +[y,[[w,x],z]] + [y,[x,[w,z]]]\end{array} &
\end{array}
\tag{5.47}
$$

The Jacobiator identity is a coherence law that should be thought of as a higher Jacobi identity for the Jacobiator.

The preceding hierarchy 'category, functor, natural transformation' together with the coherence law is entirely similar to the known hierarchy 'linear, bilinear, trilinear maps l_1, l_2, l_3' with the L_∞-conditions (a)-(e). More precisely,

Proposition 5.6.2. *There is a 1-to-1 correspondence between Leibniz 2-algebras and 2-term Leibniz infinity algebras.*

This proposition was proved in the Lie case in [BC04] and announced for the Leibniz case in [SL10]. A generalization of the latter correspondence to Lie 3-algebras and 3-term Lie infinity algebras can be found in [KMP11]. This paper allows to understand that the correspondence between higher categorified algebras and truncated infinity algebras is subject to cohomological conditions, and to see how the coherence law corresponds to the last nontrivial L_∞-condition.

The definition of Leibniz 2-algebra morphisms is God-given: such a morphism must be a functor that respects the bracket up to a natural transformation, which in turn respects the Jacobiator. More precisely,

Definition 5.6.3. Let $(L, [-,-], \mathbf{J})$ and $(L', [-,-]', \mathbf{J}')$ be Leibniz 2-algebras (in the following, we write $[-,-], \mathbf{J}$ instead of $[-,-]', \mathbf{J}'$). A *morphism* $(F, \mathbf{F})$ *of Leibniz 2-algebras* from L to L' consists of

1. a linear functor $F : L \to L'$, and
2. a bilinear natural transformation

$$\mathbf{F}_{x,y} : [Fx, Fy] \to F[x,y], \quad x, y \in L_0 \ ,$$

which make the following diagram commute

$$\begin{array}{ccc} [Fx,[Fy,Fz]] & \xrightarrow{\mathbf{J}_{Fx,Fy,Fz}} & [[Fx,Fy],Fz] + [Fy,[Fx,Fz]] \\ \downarrow{\scriptstyle [\mathbf{1}_x, \mathbf{F}_{y,z}]} & & \downarrow{\scriptstyle [\mathbf{F}_{x,y}, \mathbf{1}_z] + [\mathbf{1}_y, \mathbf{F}_{x,z}]} \\ [Fx, F[y,z]] & & [F[x,y],Fz] + [Fy, F[x,z]] \\ \downarrow{\scriptstyle \mathbf{F}_{x,[y,z]}} & & \downarrow{\scriptstyle \mathbf{F}_{[x,y],z} + \mathbf{F}_{y,[x,z]}} \\ F[x,[y,z]] & \xrightarrow{F\mathbf{J}_{x,y,z}} & F[[x,y],z] + F[y,[x,z]] \end{array} \tag{5.48}$$

Proposition 5.6.4. *There is a 1-to-1 correspondence between Leibniz 2-algebra morphisms and 2-term Leibniz infinity algebra morphisms.*

For a proof, see [BC04] and [SL10].

Composition of Leibniz 2-algebra morphisms $(F, \mathbf{F})$ is naturally given by composition of functors and whiskering of functors and natural transformations.

Proposition 5.6.5. *There is a category* `Lei2` *of Leibniz 2-algebras and morphisms.*

5.6.2 2-morphisms and their compositions

The definition of a 2-morphism is canonical:

Definition 5.6.6. Let $(F, \mathbf{F}), (G, \mathbf{G})$ be Leibniz 2-algebra morphisms from L to L'. A *Leibniz 2-algebra 2-morphism* $\boldsymbol{\theta}$ from F to G is a linear natural transformation $\boldsymbol{\theta} : F \Rightarrow G$, such that, for any $x, y \in L_0$, the following diagram commutes

$$\begin{array}{ccc} [Fx,Fy] & \xrightarrow{\mathbf{F}_{x,y}} & F[x,y] \\ \downarrow{\scriptstyle [\boldsymbol{\theta}_x, \boldsymbol{\theta}_y]} & & \downarrow{\scriptstyle \boldsymbol{\theta}_{[x,y]}} \\ [Gx,Gy] & \xrightarrow{\mathbf{G}_{x,y}} & G[x,y] \end{array} \tag{5.49}$$

Theorem 5.6.7. *There is a 1:1 correspondence between Leibniz 2-algebra 2-morphisms and 2-term Leibniz ∞-homotopies.*

Horizontal and vertical compositions of Leibniz 2-algebra 2-morphisms are those of natural transformations.

Proposition 5.6.8. *There is a strict 2-category* `Lei2Alg` *of Leibniz 2-algebras.*

Corollary 5.6.9. *The 2-categories* $\texttt{2Lei}_\infty\texttt{-Alg}$ *and* `Lei2Alg` *are 2-equivalent.*

6 Appendix

6.1 Leibniz infinity algebra

Definition 6.1.1. Lei_∞ *algebra on a graded vector space* V is given by the family of multilinear maps $l_i : V^{\otimes i} \to V$ of degrees $(i-2)$, such that for any $n > 0$ the *higher Jacobi identities* holds:

$$\sum_{i+j=n+1} \sum_{j\leqslant k\leqslant i+j-1} \sum_{\sigma\in Sh(k-j,j-1)} \varepsilon(\sigma)\cdot \mathrm{sign}(\sigma)\cdot(-1)^{(i-k+j)(j-1)}\cdot(-1)^{j(v_{\sigma(1)}+\ldots+v_{\sigma(k-j)})}\times \\ \times l_i(v_{\sigma(1)},...,v_{\sigma(k-j)},l_j(v_{\sigma(k+1-j)},...,v_{\sigma(k-1)},v_k),v_{k+1},...,v_{i+j-1})=0, \tag{6.1}$$

Lei_∞ algebras are nonsymmetric versions of L_∞ algebras, that is if we additionally impose the graded antisymmetry condition on the higher brackets $\{l_i\}$ we will get the L_∞ algebra.

Theorem 6.1.2. Lei_∞ *algebra over a finite dimensional graded vector space* V *is given by the differential* d *on the quasi-free DGZA* $\mathrm{Zin}(s^{-1}V^*)$ *or dually by the codifferential* $D = d^*$ *on the coalgebra* $\mathrm{Zin}^c(sV)$ *. And the condition* $d^2 = 0$ *(dually* $D^2 = 0$*) encodes the bunch of higher Jacobi identities (6.1).*

Proof. Consider an arbitrary finite dimensional graded vector space W and the differential d on the quasi-free DGZA $\mathrm{Zin}(W)$. The differential d is given by its action on the generators, that is by the action on the vector space W. By applying the Leibniz rule one can get the action of the differential on the whole space $\mathrm{Zin}(W)$ knowing only its action on W. We define the components of the differential:

$$d_k : W \to W^{\otimes k} \subset \mathrm{Zin}(W),\ d|_W = d_1 + d_2 + ...$$

Then for any element $w_1...w_p \in W^{\otimes p} \subset \mathrm{Zin}(W)$ we have

$$d(w_1...w_p) = d((((w_1\cdot w_2)\cdot w_3)...)\cdot w_p) = \sum_{i=1}^{p}(-1)^{|w_1|+...+|w_{i-1}|}((((w_1\cdot w_2)...dw_i)\cdot w_{i+1})...)\cdot w_p) = \\ = \sum_{i=1}^{p}(-1)^{|w_1|+...+|w_{i-1}|}(((((w_1w_2...w_{i-1})\cdot \sum_{k>0} d_k w_i)\cdot w_{i+1})...)\cdot w_p) = \\ = \sum_{i=1}^{p}\sum_{k>0}\sum_{\sigma\in Sh(i-1,k-1)}(\sigma^{-1}\otimes \mathrm{id}^{(p-i+1)})(\mathrm{id}^{\otimes(i-1)}\otimes d_k\otimes \mathrm{id}^{\otimes(p-i)})(w_1w_2...w_p). \tag{6.2}$$

If for an arbitrary -1 degree derivation d on $\mathrm{Zin}(W)$ the action of d^2 on generators is zero then it is zero on the whole space $\mathrm{Zin}(W)$, and therefore the derivation becomes a differential. It follows from the following identity:

$$d^2(w\cdot v) = d^2 w\cdot v + w\cdot d^2 v = 0.$$

The condition that $d^2 = 0$ reads as follows:

$$d(dw) = \sum_{p>0} dd_p w \overset{(6.2)}{=} \sum_{p>0}\sum_{\substack{0<i\leqslant p\\ k>0}}\sum_{\sigma\in Sh(i-1,k-1)}(\sigma^{-1}\otimes \mathrm{id}^{(p-i+1)})(\mathrm{id}^{\otimes(i-1)}\otimes d_k\otimes \mathrm{id}^{\otimes(p-i)})d_p w \\ = \sum_{n>0}\sum_{\substack{0<i\leqslant p\\ k+p-1=n}}\sum_{\sigma\in Sh(i-1,k-1)}(\sigma^{-1}\otimes \mathrm{id}^{(p-i+1)})(\mathrm{id}^{\otimes(i-1)}\otimes d_k\otimes \mathrm{id}^{\otimes(p-i)})d_p w.$$

In the dual language, the transposed map $D = d^*$ is a codifferential on the coalgebra $\mathrm{Zin}^c(W^*)$. And the transposition of the last identity will encode that $D^2 = 0$:

$$\sum_{n>0} \sum_{\substack{0<i\leqslant p \\ k+p-1=n}} \sum_{\sigma \in Sh(i-1,k-1)} D_p(\mathrm{id}^{\otimes(i-1)} \otimes D_k \otimes \mathrm{id}^{\otimes(p-i)})(\sigma \otimes \mathrm{id}^{(p-i+1)}) = 0, \qquad (6.3)$$

where the transposed maps $D_p = d_p^* : W^{*\otimes p} \to W^*$ called the corestrictions of the codifferential D.

In equation (6.3) the weight of the operator inside the sum $\sum\limits_{n>0}$ is equal to $2-p-k = 1-n$, so it depends only on n, i.e. the last equation splits into the series of equations:

$$(D^2)_n = \sum_{\substack{0<i\leqslant p \\ k+p-1=n}} \sum_{\sigma \in Sh(i-1,k-1)} D_p(\mathrm{id}^{\otimes(i-1)} \otimes D_k \otimes \mathrm{id}^{\otimes(p-i)})(\sigma \otimes \mathrm{id}^{(p-i+1)}) = 0, \quad \text{where } n > 0.$$

So the condition $D^2 = 0$ on the coalgebra $\mathrm{Zin}^c(sV)$ reads as follows:

$$\sum_{\substack{0<i\leqslant p \\ k+p-1=n}} \sum_{\sigma \in Sh(i-1,k-1)} D_p(\mathrm{id}^{\otimes(i-1)} \otimes D_k \otimes \mathrm{id}^{\otimes(p-i)})(\sigma \otimes \mathrm{id}^{(p-i+1)})(sv_1...sv_n) = 0, \quad n > 0.$$

Now we insert identities of the type $(-1)^{\frac{i(i-1)}{2}} s^{\otimes i}(s^{-1})^{\otimes i} = \mathrm{id}^{\otimes i}$ in two places:

$$\sum_{\substack{0<i\leqslant p \\ k+p-1=n}} \overbrace{s^{-1} D_p s^{\otimes p}}^{l_p} (-1)^{\frac{p(p-1)}{2}} \overbrace{(s^{-1})^{\otimes p} \left(\mathrm{id}^{\otimes(i-1)} \otimes D_k \otimes \mathrm{id}^{\otimes(p-i)}\right) s^{\otimes n}}^{\pm \mathrm{id}^{\otimes(i-1)} \otimes l_k \otimes \mathrm{id}^{\otimes(p-i)}} (-1)^{\frac{n(n-1)}{2}} \circ$$
$$\circ \underbrace{(s^{-1})^{\otimes n} \sum_{\sigma \in Sh(i-1,k-1)} \left(\sigma \otimes \mathrm{id}^{(p-i+1)}\right) s^{\otimes n}}_{\left(\sum\limits_{\sigma \in Sh(i-1,k-1)} \pm \sigma \otimes \mathrm{id}^{(p-i+1)}\right)} = 0. \qquad (6.4)$$

The precise signs in the formula above are the following:

1. $s^{-1} D_p s^{\otimes p} = l_p$,
2. $(s^{-1})^{\otimes p} \left(\mathrm{id}^{\otimes(i-1)} \otimes D_k \otimes \mathrm{id}^{\otimes(p-i)}\right) s^{\otimes n} = (-1)^{[(i-1)+\frac{p(p-1)}{2}+(p-i)k]} \cdot \mathrm{id}^{\otimes(i-1)} \otimes l_k \otimes \mathrm{id}^{\otimes(p-i)}$,
3. $(s^{-1})^{\otimes n} \sum\limits_{\sigma \in Sh(i-1,k-1)} \left(\sigma \otimes \mathrm{id}^{(p-i+1)}\right) s^{\otimes n} =$

   $= \sum\limits_{\sigma \in Sh(i-1,k-1)} \left((-1)^{\frac{n(n-1)}{2}} \cdot sign(\sigma) \cdot \sigma \otimes \mathrm{id}^{(p-i+1)}\right)$.

Now we change the indices of summation $p \to i$, $k \to j$, $i \to (k-j+1)$, to be better in keeping with the results in the literature, we get for each $n > 0$:

$$\sum_{i+j=n+1} \sum_{j \leqslant k \leqslant i+j-1} \sum_{\sigma \in Sh(k-j,j-1)} (-1)^{(i-k+j)(j-1)} \cdot \mathrm{sign}(\sigma)$$
$$l_i \left(\mathrm{id}^{\otimes(k-j)} \otimes l_j \otimes \mathrm{id}^{\otimes(i-k+j-1)}\right) \left(\sigma \otimes \mathrm{id}^{\otimes(i-k+j)}\right) = 0 \qquad (6.5)$$

One can expand out the condensed tensor notation and get (6.1). □

6.2 Leibniz infinity algebra morphism

Definition 6.2.1. The morphism between two Lei_∞ algebras over V and W is given by the family of multilinear maps $\varphi_i : V^{\otimes i} \to W$ of degree $(i-1)$ which satisfy the following identities:

$$\sum_{i=1}^{n} \sum_{k_1+...+k_i=n} \sum_{\sigma\in Hsh(k_1,...,k_i)} (-1)^{\sum\limits_{r=1}^{i-1}(i-r)k_r+\frac{i(i-1)}{2}} \cdot (-1)^{\sum\limits_{r=2}^{i}(k_r-1)(v_{\sigma(1)}+...+v_{\sigma(k_1+...+k_{r-1})})} \times$$
$$\times \varepsilon(\sigma)\cdot \mathrm{sign}(\sigma)\cdot l_i\left(\varphi_{k_1}(v_{\sigma(1)},...,v_{\sigma(k_1)}),...,\varphi_{k_i}(v_{\sigma(k_1+...+k_{i-1}+1)},...,v_{\sigma(k_1+...+k_i)})\right) =$$
$$= \sum_{i+j=n+1} \sum_{j\leqslant k\leqslant i+j-1} \sum_{\sigma\in Sh(k-j,j-1)} (-1)^{k+(i-k+j)j}\cdot(-1)^{j(v_{\sigma(1)}+...+v_{\sigma(k-j)})}\cdot \varepsilon(\sigma)\cdot \mathrm{sign}(\sigma)\times$$
$$\times \varphi_i(v_{\sigma(1)},...,v_{\sigma(k-j)}, l_j(v_{\sigma(k+1-j)},...,v_{\sigma(k-1)},v_k),v_{k+1},...,v_{i+j-1}). \tag{6.6}$$

Theorem 6.2.2. Lei_∞*-algebra morphism between* Lei_∞*-algebras over* V *and* W *is given by the differential graded algebra homomorphism* $f : \mathrm{Zin}(s^{-1}W^*) \to \mathrm{Zin}(s^{-1}V^*)$*, or dually by the graded differential coalgebra homomorphism* $F = f^* : \mathrm{Zin}^c(sV) \to \mathrm{Zin}^c(sW)$*. The morphism condition* $fd - df = 0$ *(or dually* $FD - DF = 0$*) encodes the bunch of higher identities (6.6).*

Proof. Consider an arbitrary graded vector spaces U and U'. An arbitrary algebra homomorphism $f : \mathrm{Zin}(U) \to \mathrm{Zin}(U')$ is given by its action on generators:

$$f(u_1...u_p) = f((((u_1\cdot u_2)\cdot u_3)...)\cdot u_p) = ((((fu_1\cdot fu_2)\cdot fu_3)...)\cdot fu_p) =$$
$$= \sum_{k_1+...+k_p=p}^{\infty} ((((f_{k_1}u_1\cdot f_{k_2}u_2)\cdot f_{k_3}u_3)...)\cdot f_{k_p}u_p) =$$
$$= \sum_{k_1+...+k_p=p}^{\infty} \left(\sum_{\sigma\in Sh(k_1,k_2-1)} \sigma^{-1}\otimes \mathrm{id}^{\otimes(1+k_3+...+k_p)}\right)...\left(\sum_{\sigma\in Sh(k_1+...+k_{p-1},k_p-1)} \sigma^{-1}\otimes \mathrm{id}\right)\circ$$
$$\circ\left(f_{k_1}\otimes...\otimes f_{k_p}\right)(u_1...u_p) =$$
$$= \sum_{k_1+...+k_p=p}^{\infty} \sum_{\sigma\in Hsh(k_1,...,k_p)} \sigma^{-1}\left(f_{k_1}\otimes...\otimes f_{k_p}\right)(u_1...u_p), \tag{6.7}$$

where the the maps $f_k : U \to U'^{\otimes k}$ are the components of the restriction of the homomorphism f on the U:

$$f|_U = f_1 + f_2 + ...$$

If for the algebra morphism f the condition $df - fd = 0$ holds on the generators then it also holds on the whole space $\mathrm{Zin}(U)$, it follows from the fact that

$$(fd-df)(u\cdot w) = (fd-df)u\cdot w + (-1)^{\overline{u}}u\cdot(fd-df)w.$$

Then the condition that the algebra homomorphism f is a *differential* algebra homomorphism reads as follows:

$$(fd-df)u=\sum_{p>0} fd_pu-\sum_{p>0} df_pu \overset{(6.2)\&(6.7)}{=}$$

$$= \sum_{n>0}\left[\sum_{\substack{k_1+...+k_p=n\\0<p\leqslant n}}\ \sum_{\sigma\in Hsh(k_1,...,k_p)} \sigma^{-1}\left(f_{k_1}\otimes ... \otimes f_{k_p}\right)d_pu- \right.$$
$$\left. -\sum_{\substack{0<i\leqslant p\\k+p-1=n}}\ \sum_{\sigma\in Sh(i-1,k-1)} (\sigma^{-1}\otimes \mathrm{id}^{(p-i+1)})(\mathrm{id}^{\otimes(i-1)}\otimes d_k\otimes \mathrm{id}^{\otimes(p-i)})f_pu\right]=0. \tag{6.8}$$

In the dual language, the transposed map $F=f^*$ is a homomorphism of the coalgebras $F:\mathrm{Zin}^c(U'^*)\to \mathrm{Zin}^c(U^*)$. And the transposition of the last identity encodes that $FD-DF=0$:

$$\sum_{n>0}\left[\sum_{\substack{k_1+...+k_p=n\\0<p\leqslant n}}\ \sum_{\sigma\in Hsh(k_1,...,k_p)} D_p\left(F_{k_1}\otimes ... \otimes F_{k_p}\right)\sigma- \right.$$
$$\left. -\sum_{\substack{0<i\leqslant p\\k+p-1=n}}\ \sum_{\sigma\in Sh(i-1,k-1)} F_p(\mathrm{id}^{\otimes(i-1)}\otimes D_k\otimes \mathrm{id}^{\otimes(p-i)})(\sigma\otimes \mathrm{id}^{(p-i+1)})\right]=0, \tag{6.9}$$

where the transposed maps $F_k=f_k^*:S^k(U'^*)\to U^*$ are called the corestrictions of the morphism F.

In equation (6.9) the weight of the operator inside the sum $\sum\limits_{n=1}^{\infty}$ is equal to $1-n$, so it depends only on n, that is the last equation splits into the series of equations:

$$\sum_{\substack{k_1+...+k_p=n\\0<p\leqslant n}}\ \sum_{\sigma\in Hsh(k_1,...,k_p)} D_p\left(F_{k_1}\otimes ... \otimes F_{k_p}\right)\sigma-$$
$$-\sum_{\substack{0<i\leqslant p\\k+p-1=n}}\ \sum_{\sigma\in Sh(i-1,k-1)} F_p(\mathrm{id}^{\otimes(i-1)}\otimes D_k\otimes \mathrm{id}^{\otimes(p-i)})(\sigma\otimes \mathrm{id}^{(p-i+1)})=0.$$

Now we insert in certain places the identities of the type $(-1)^{\frac{i(i-1)}{2}} s^{\otimes i}(s^{-1})^{\otimes i} = \mathrm{id}^{\otimes i}$ and get:

$$
\begin{aligned}
&\sum_{p=1}^{n} \sum_{k_1+\ldots+k_p=n} \overbrace{s^{-1} D_p s^{\otimes p}}^{l_p} (-1)^{\frac{p(p-1)}{2}} \overbrace{(s^{-1})^{\otimes p} \left(F_{k_1} \otimes \ldots \otimes F_{k_p}\right) s^{\otimes n}}^{\pm \varphi_{k_1} \otimes \ldots \otimes \varphi_{k_p}} \circ \\
&\circ (-1)^{\frac{n(n-1)}{2}} \overbrace{\sum_{\sigma \in Hsh(k_1,\ldots,k_p)} (s^{-1})^{\otimes n} \sigma s^{\otimes n}}^{\sum_{\sigma \in Hsh(k_1,\ldots,k_p)} \pm \sigma} = \\
&= \sum_{k+p-1=n} \sum_{0<i\leqslant p} \overbrace{s^{-1} F_p s^{\otimes p}}^{\varphi_p} (-1)^{\frac{p(p-1)}{2}} \overbrace{(s^{-1})^{\otimes p} (\mathrm{id}^{\otimes(i-1)} \otimes D_k \otimes \mathrm{id}^{\otimes(p-i)}) s^{\otimes n}}^{\pm(\mathrm{id}^{\otimes(i-1)} \otimes l_k \otimes \mathrm{id}^{\otimes(p-i)})} \circ \\
&\circ (-1)^{\frac{n(n-1)}{2}} \overbrace{\sum_{\sigma \in Sh(i-1,k-1)} (s^{-1})^{\otimes n} (\sigma \otimes \mathrm{id}^{(p-i+1)}) s^{\otimes n}}^{\sum_{\sigma \in Sh(i-1,k-1)} \pm \sigma \otimes \mathrm{id}^{(p-i+1)}}.
\end{aligned}
\tag{6.10}
$$

The precise signs in the formula above are the following:

1. $s^{-1} D_p \, (s)^{\otimes p} = l_p$,
2. $(s^{-1})^{\otimes p} \left(F_{k_1} \otimes \ldots \otimes F_{k_p}\right) s^{\otimes n} = (-1)^{\left[\sum_{r=1}^{p-1} (p-r) k_r\right]} \cdot \varphi_{k_1} \otimes \ldots \otimes \varphi_{k_p}$,
3. $\sum_{\sigma \in Hsh(k_1,\ldots,k_p)} (s^{-1})^{\otimes n} \sigma s^{\otimes n} = \sum_{\sigma \in Hsh(k_1,\ldots,k_p)} (-1)^{\frac{n(n-1)}{2}} \cdot \mathrm{sign}(\sigma) \cdot \sigma$,
4. $(s^{-1}) F_p s^{\otimes p} = \varphi_p$,
5. $(s^{-1})^{\otimes p} \left(\mathrm{id}^{\otimes(i-1)} \otimes D_k \otimes \mathrm{id}^{\otimes(p-i)}\right) s^{\otimes n} = (-1)^{[(i-1) + \frac{p(p-1)}{2} + (p-i)k]} \cdot \mathrm{id}^{\otimes(i-1)} \otimes l_k \otimes \mathrm{id}^{\otimes(p-i)}$,
6. $\sum_{\sigma \in Sh(i-1,k-1)} (s^{-1})^{\otimes n} \left(\sigma \otimes \mathrm{id}^{(p-i+1)}\right) s^{\otimes n} = \sum_{\sigma \in Sh(i-1,k-1)} \left((-1)^{\frac{n(n-1)}{2}} \cdot \mathrm{sign}(\sigma) \cdot \sigma \otimes \mathrm{id}^{(p-i+1)}\right)$.

We evaluate the operators (6.10) on the elements $v_1 \otimes \ldots \otimes v_n$ and get identities (6.6). □

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